Routing and Treewidth
(Recent Developments)

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Routing Problems

Undir graph $G=(V,E)$ and k node-pairs $s_1t_1, s_2t_2, \ldots, s_kt_k$

EDP: can all pairs be connected by edge-disjoint paths?

NDP: can pairs be connected by node-disjoint paths?

ANF: is there a multiflow for pairs s.t each pair has flow of one unit?

Hardness of Decision Versions

NDP/EDP NP-Complete if k is part of input [Karp, Even-Itai-Shamir]

EDP/NDP in P if k is fixed [Robertson-Seymour'90]

ANF is in P via LP

Maximum Throughput Problems

Undir graph $G=(V,E)$ and k node-pairs $s_1t_1, s_2t_2, \ldots, s_kt_k$

MEDP: max # of pairs connected by edge-disj paths

MNDP: max # of pairs connected by node-disj paths

MANF: max # of pairs with flow of one unit

Hardness

Undir graph $G=(V,E)$ and k node-pairs s_1t_1, s_2t_2, …, s_kt_k

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MANF: max # of pairs with flow of one unit

MEDP/MNDP/MANF are $\Omega(\log^{1/2-\epsilon} n)$-hard [Andrews-etal'05]

Approximation

Undir graph $G=(V,E)$ and k node-pairs s_1t_1, s_2t_2, …,s_kt_k

MEDP: max # of pairs connected by edge-disj paths

MNDP: max # of pairs connected by node-disj paths

ANF: max # of pairs with flow of one unit

MEDP/MNDP have $O(n^{1/2})$-approx [C-Khanna-Shepherd, Kolliopoulos-Stein]

ANF has $O(\log^2 n)$ –approx [C-Khanna-Shepherd'05]

Obvious Open Problem

MEDP/MNDP/MANF are $\Omega(\log^{1/2-\epsilon} n)$-hard

MEDP/MNDP have $O(n^{1/2})$-approx

Close the gap!

No good relaxation for **MEDP/MNDP**

Multicommodity Flow Relaxation

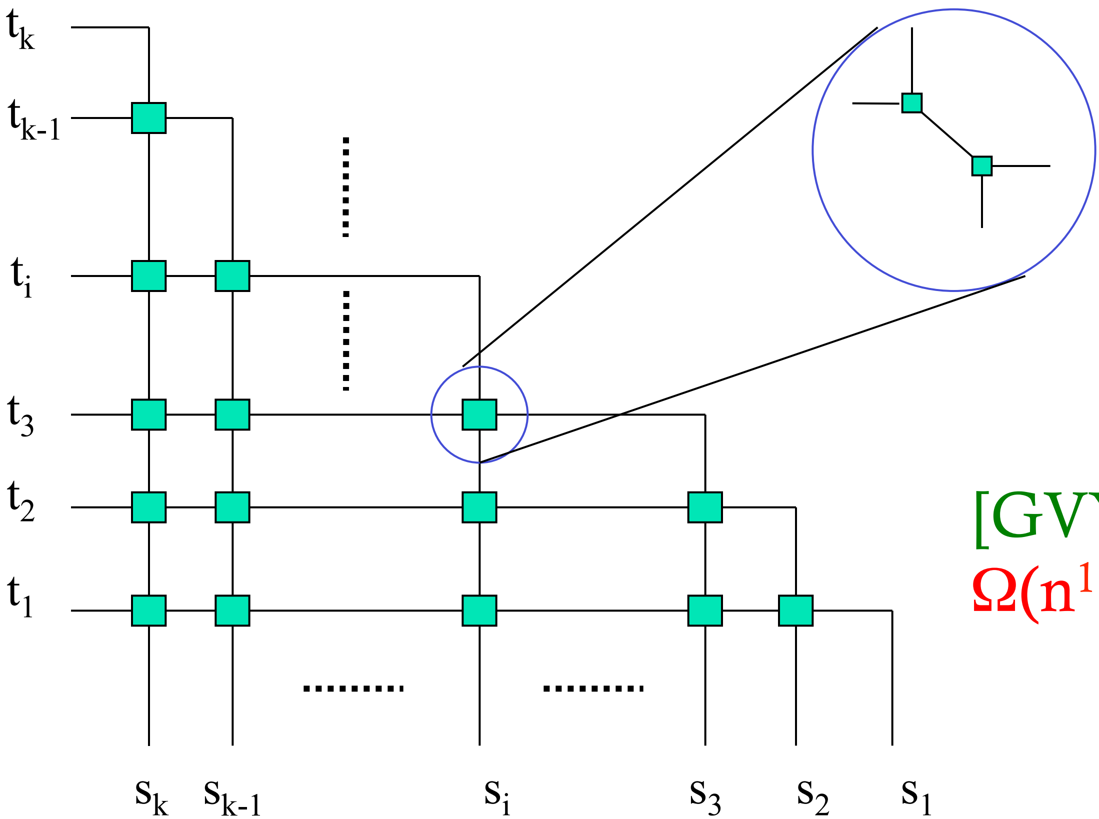
variable x_i for each pair $s_i t_i$

$\max \sum x_i$ s.t

G supports multicomm. flow of x_i for pair $s_i t_i$

$0 \leq x_i \leq 1$

Integrality Gap for MEDP/ MNDP



[GVY]

$\Omega(n^{1/2})$ gap

Routing with Congestion

Question: Is integrality gap small with congestion c?

$O(1)$ gap with $O(\log n/\log \log n)$ congestion via randomized rounding [Raghavan-Thompson'87]

Main interest: constant c and $c = 2$

Recent Results

[Chuzhoy'11] polylog(k) integrality gap for **MEDP** with congestion 14

[Chuzhoy-Li'12] polylog(k) integrality gap for **MEDP** with congestion 2

[C-Ene'13] polylog(k) integrality gap for **MNDP** with congestion 51

Work in Progress

[C-Chuzhoy'14]

polylog(k) integrality gap for **MNDP** with congestion 2

Corollary of stronger result

polylog(k) integrality gap for **ANF** with node-capacities where unit flow for routed pair is *half-integral*

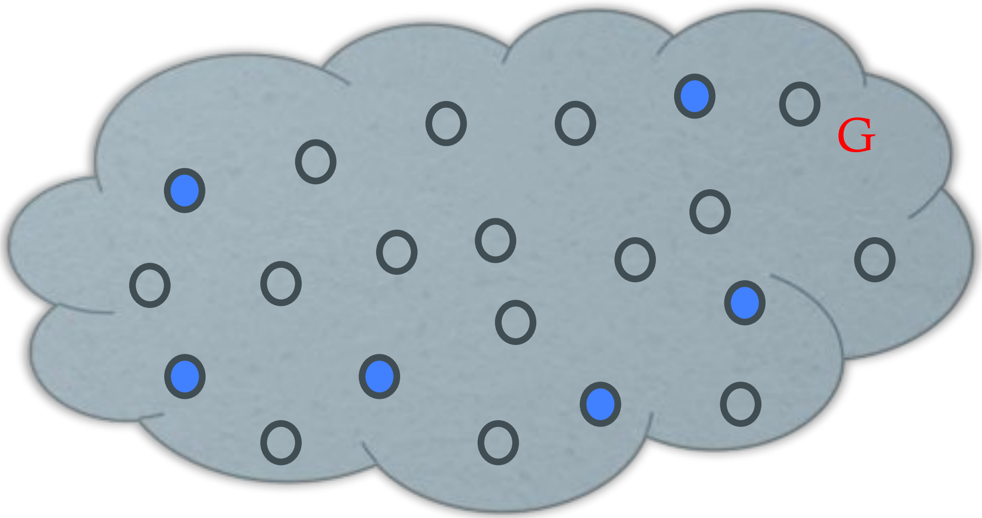
Best we can hope for via flow relaxation

Rest of Talk

- Results follow from structural and algorithmic results on treewidth and well-linked sets
- Outline connections and recent results on treewidth

Well-linked Sets

A set $X \subseteq V$ is **well-linked** in G if for all $A, B \subseteq X$ there are $\min(|A|, |B|)$ node-disjoint A-B paths



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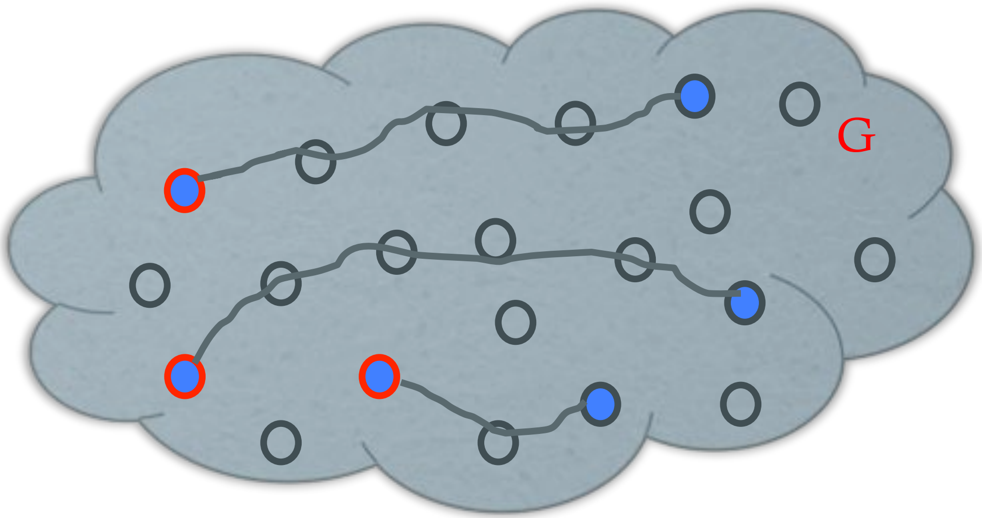

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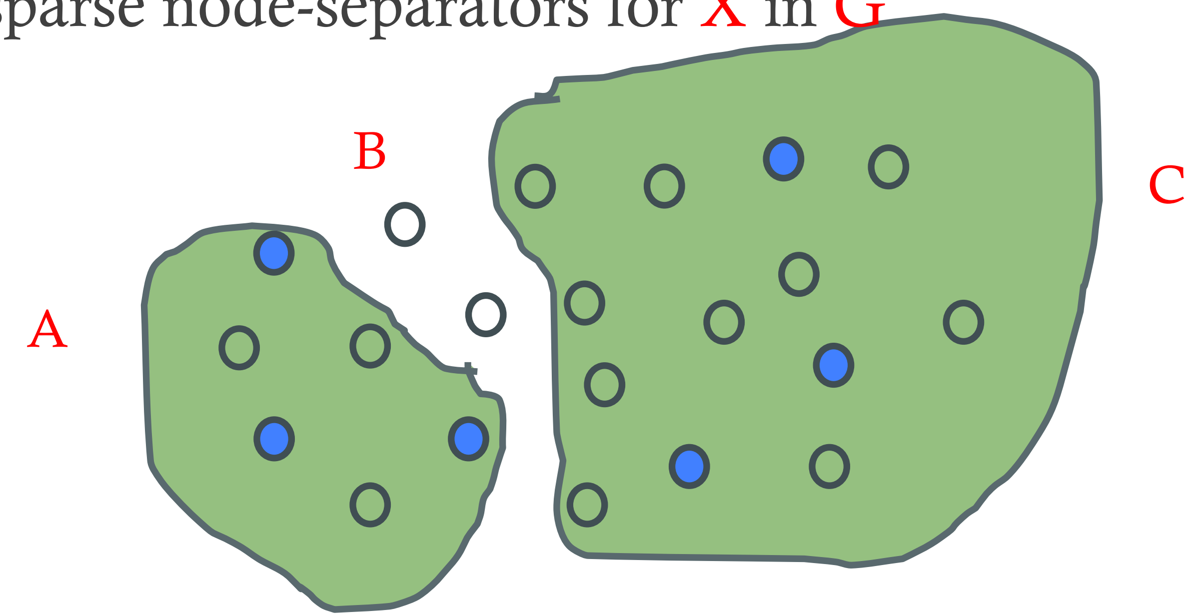
No sparse node-separators for X in G



Treewidth & Well-linked Sets

A set $X \subseteq V$ is **well-linked** in G if for all $A, B \subseteq X$ there are $\min(|A|, |B|)$ node-disjoint A-B paths

No sparse node-separators for X in G

$wl(G)$ = cardinality of the largest well-linked set in G

$tw(G)$ = treewidth of G

$$wl(G) \leq tw(G) \leq 4\, wl(G)$$

Treewidth and Routing

[C-Khanna-Shepherd'05]

Reduce integrality gap question to *well-linked instances* with polylog(k) loss

$G=(V,E)$ and terminal set $X = \{s_1,t_1,\ldots,s_k,t_k\}$

X is well-linked in G

Treewidth and Routing

[C-Khanna-Shepherd'05]

Reduce integrality gap question to *well-linked instances* with polylog(k) loss

$G=(V,E)$ and terminal set $X = \{s_1,t_1,\ldots,s_k,t_k\}$

X is well-linked in G

Question: If $tw(G) = k$ does G have a large routing structure?

Treewidth and Routing

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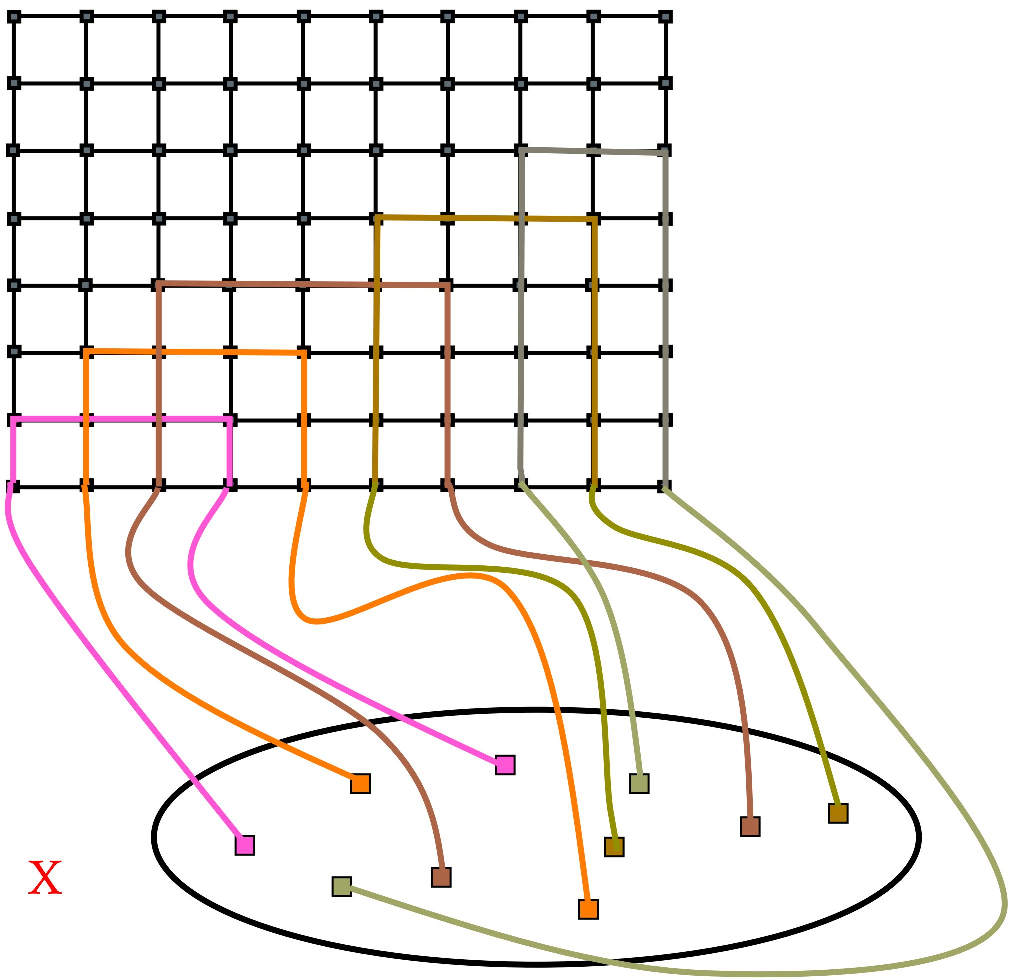
[Robertson-Seymour-Thomas] If $tw(G) = k$ and G is *planar* then G has a grid-minor of size $\Omega(k)$

Grid minors are good routing structures.


X

Treewidth and Routing

[Rao-Zhou'08] Idea for general graphs:

"Embed" an *expander* using cut-matching game of [Khandekar-Rao-Vazirani'05]

Treewidth and Routing

[Chuzhoy'11,Chuzhoy-Li'12]

If $tw(G) \geq k$ then there is an expander of size $k/polylog(k)$ that can be "embedded" into G with edge congestion 2

[C-Ene'13]

If $tw(G) \geq k$ then there is an expander of size $k/polylog(k)$ that can be "embedded" into G with node congestion 51

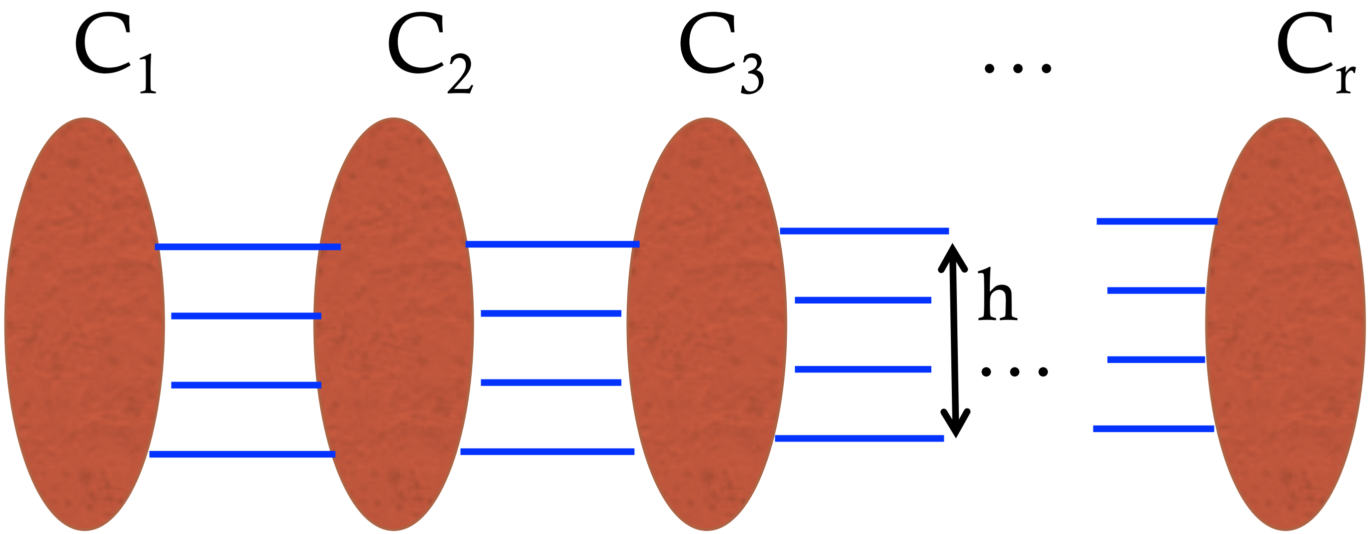
[C-Chuzoy'14] *improve node congestion to 2*

A Structural Result on Treewidth

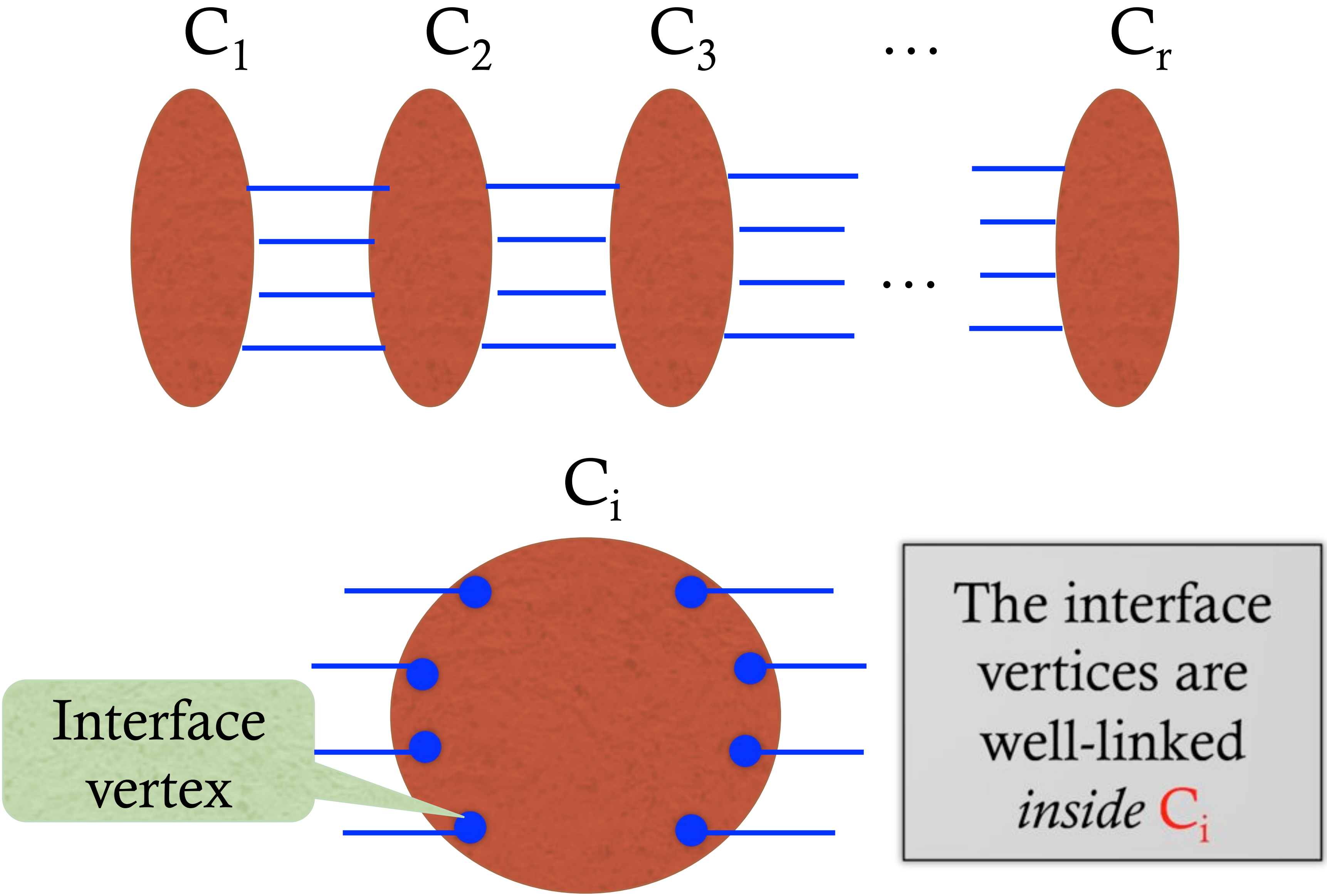
[C-Chuzhoy'13]

Informal: Every graph has a *path-of-sets-system* as a subgraph and whose size depends on the treewidth.

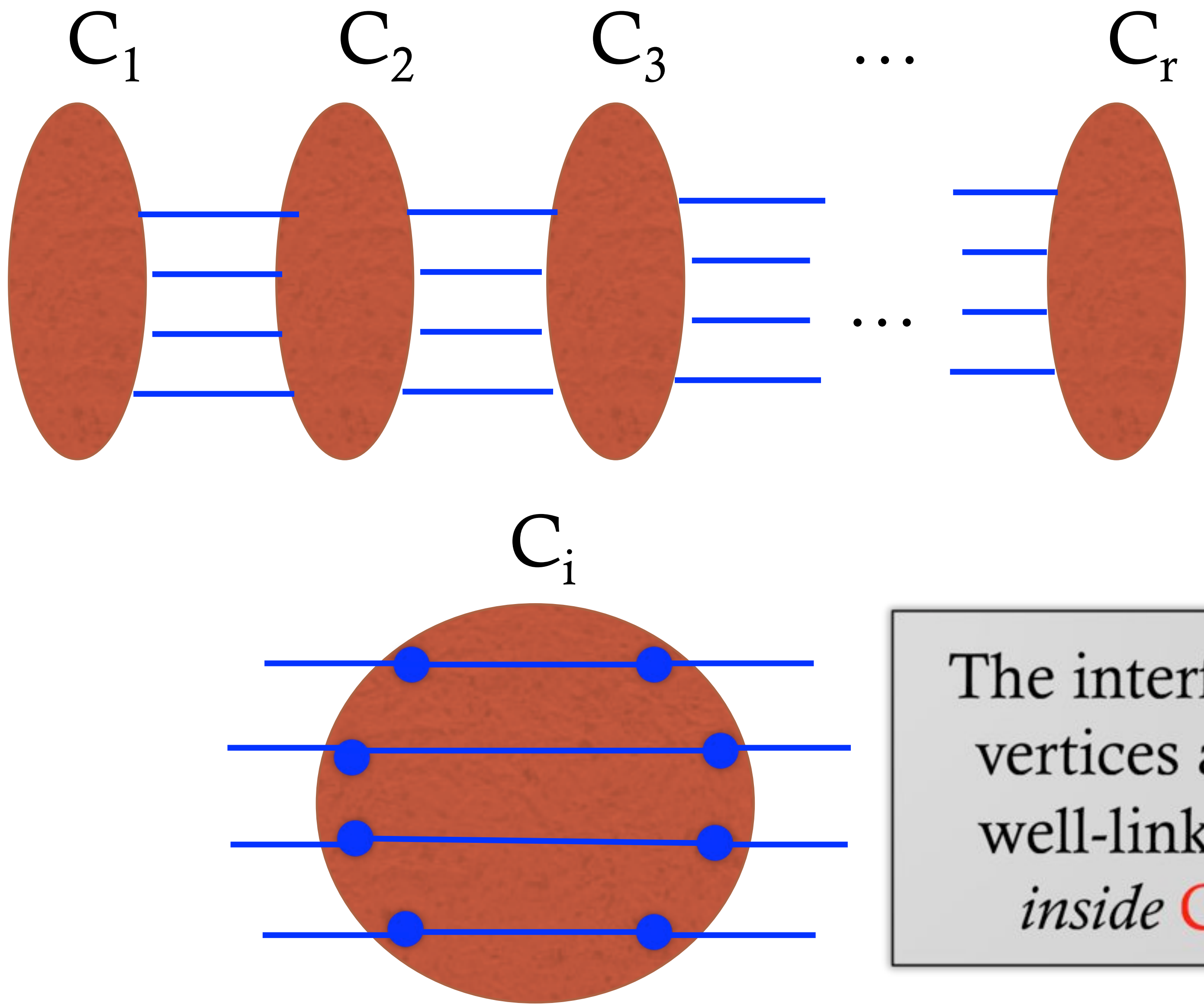
Path-of-Sets System



- Each C_i is a connected cluster
- The clusters are disjoint
- Every consecutive pair of clusters connected by h paths
- All blue paths are disjoint from each other and internally disjoint from the clusters



C_1
C_2
C_3
...
C_r
...
C_i
Interface vertex
The interface vertices are well-linked *inside* C_i



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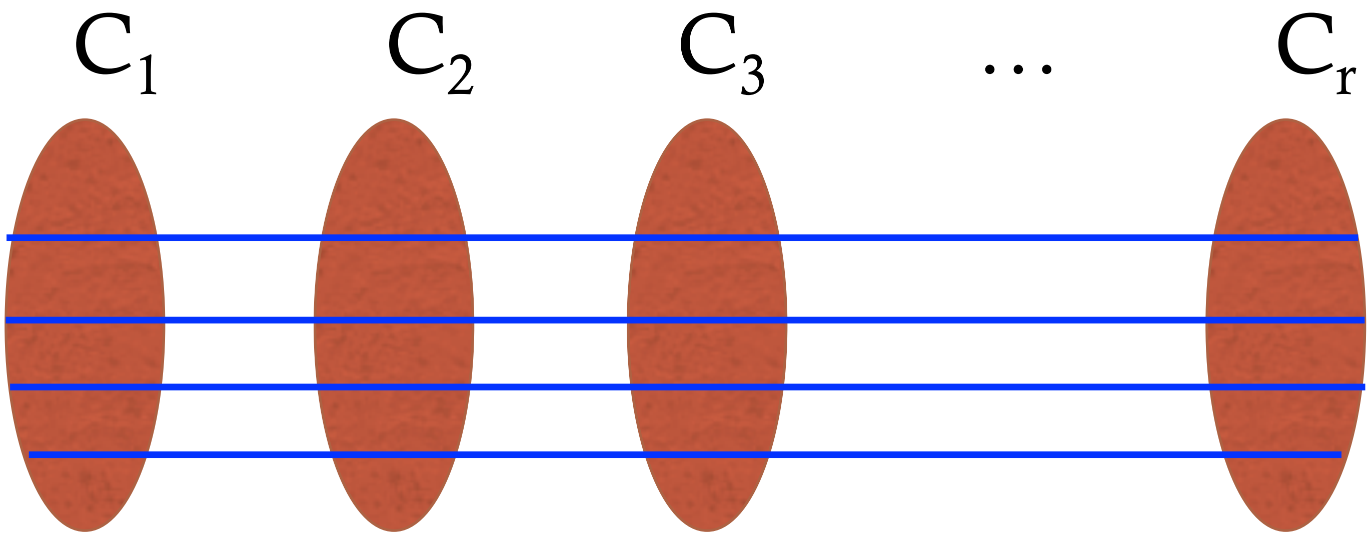

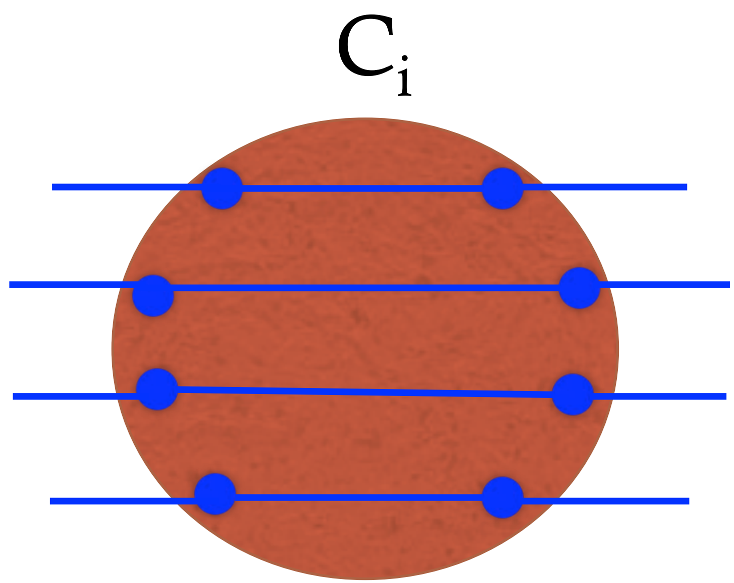

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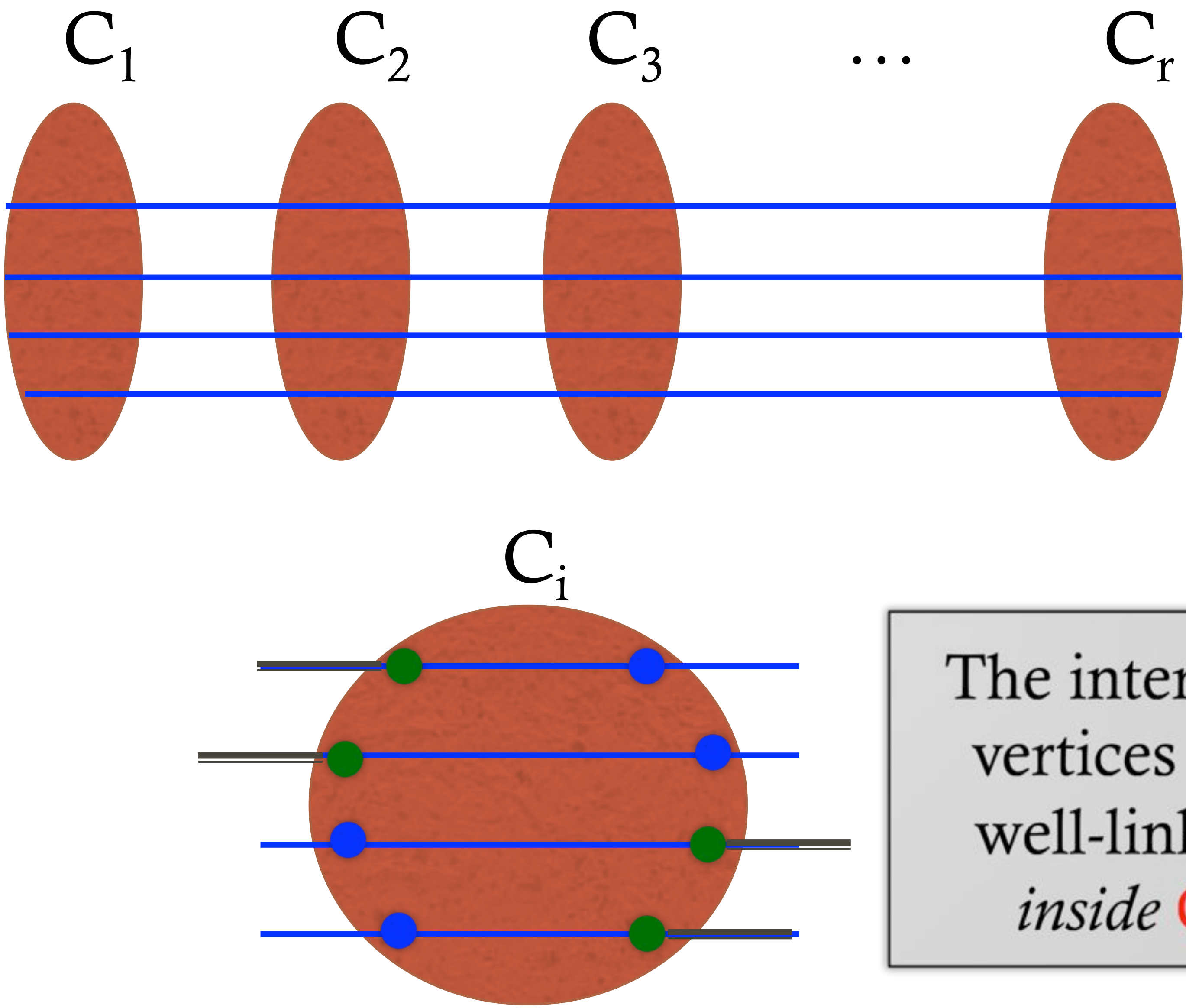

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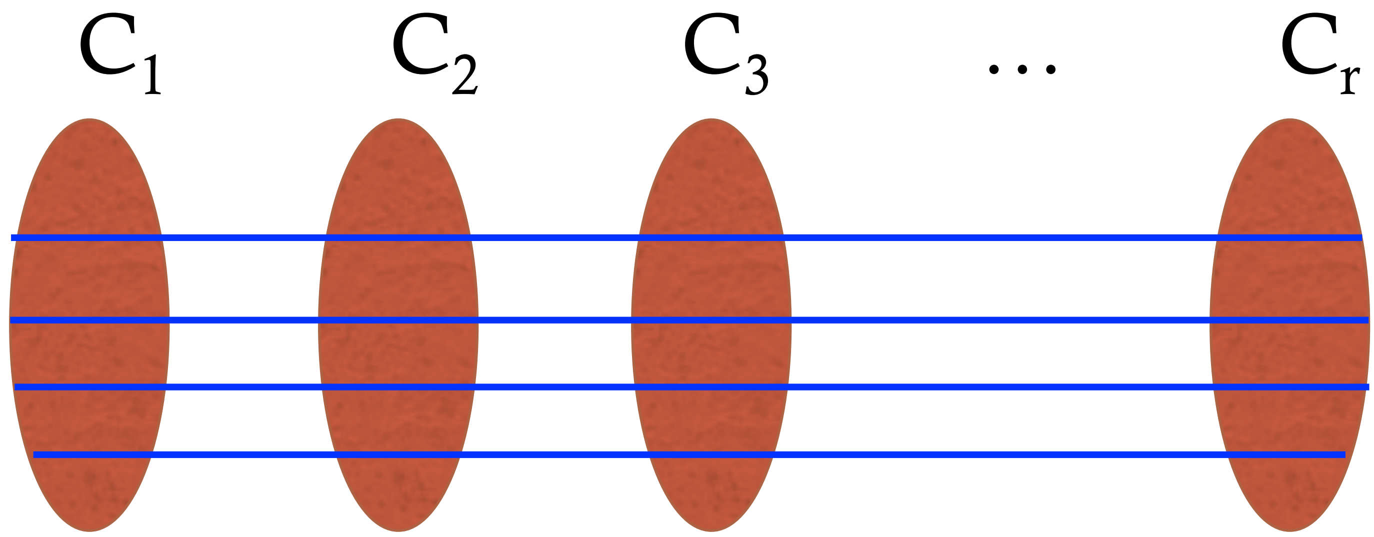
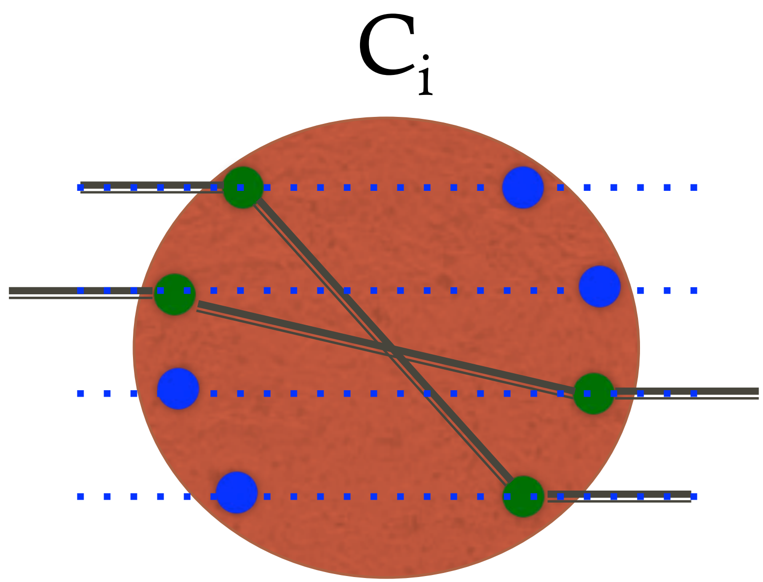




The interface vertices are well-linked *inside* C_i

A Structural Result on Treewidth

[C-Chuzhoy'13]

Theorem: If $tw(G) \geq k$ and $h\ r^{19} \leq k/\mathrm{polylog}(k)$ then there G has a path-of-sets systems with parameters h, r.

Moreover, a poly-time algorithm to construct it.

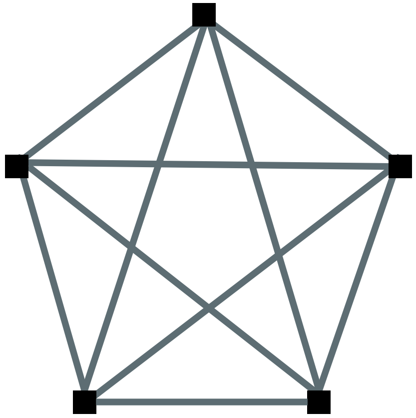
Applications

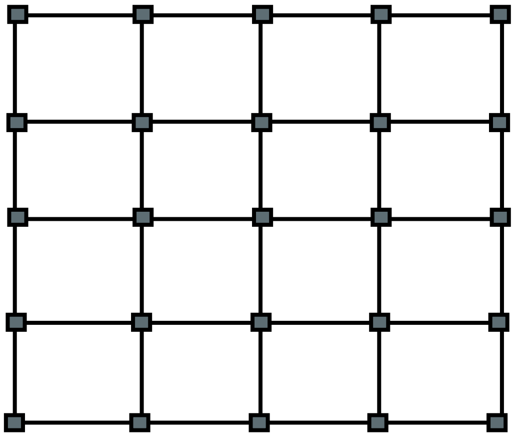
From theorem on path-of-sets and additional ideas:

- Polynomial-sized grid-minor theorem
- Embedding expander of size $k/\text{polylog}(k)$ with node-congestion 2 via cut-matching game
- Treewidth sparsifier

Robertson-Seymour Grid-Minor Theorem(s)

Theorem: tw(G) $\geq$ f(k) implies G contains a clique minor of size k or a grid minor of size k





Robertson-Seymour Grid-Minor Theorem(s)

Theorem: $tw(G) \geq f(k)$ implies G contains a clique minor of size k or a grid minor of size k

Corollary (Grid-minor theorem): $tw(G) \geq f(k)$ implies G contains a grid minor of size k

Previous best bound: $f(k) = 2^{O(k^2 \log k)}$ [Kawarabayashi-Kobayashi'12, Leaf-Seymour'12]

$tw(G) \geq h$ implies grid minor of size at least $(\log h)^{1/2}$

Improved Grid-Minor Thm

[C-Chuzhoy'13]

Theorem: $tw(G) \geq k$ implies that G has a grid-minor of size $\Omega(k^{\delta})$ where $\delta \geq 1/98 - o(1)$

Treewidth Sparsifier

[C-Chuzhoy'14]

Let $tw(G) = k$. G has a subgraph H such that

- $tw(H) \geq k/\text{polylog}(k)$
- max deg of H is 3
- # of deg 3 nodes in H is $O(k^4)$

Poly-time algorithm to construct H given G

Directed Graphs

Dir graph MEDP is $\Omega(n^{1/2-\epsilon})$ hard [Guruswami etal]

Even with congestion c problem is $n^{\Omega(c)}$ – hard [Chuzhoy-Guruswami-Khanna-Talwar]

Symmetric demands:

1. pairs are unordered
2. $s_i t_i$ routed if (s_i,t_i) *and* (t_i,s_i) are routed

Directed Graphs with Symmetric Demands

Symmetric demands:

1. pairs are unordered
2. $s_i t_i$ routed if (s_i, t_i) *and* (t_i, s_i) are routed

Conjecture: polylog(k) approx with constant congestion via LP feasible for MEDP/MNDP

[C-Ene'13] polylog(k) integrality gap/approx for ANF with symmetric demands with O(1) congestion

Connection to *directed treewidth*

Directed Treewidth

Question: If G has *directed treewidth k* does it have a routing structure of size k/polylog(k) ?

Open Problems

- Improve Grid-Minor theorem parameters
- Similar results for *directed* treewidth?
- Several “lower-level” research questions

Thank You!