Independent Sets in Elimination Graphs: Submodular Objective

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Max Weight Independent Set

$G = (V, E)$ undirected graph

$S \subseteq V$ is an independent/stable set if G[S] has no edges

Problem: given G and weights $w(v), v \in V$

find max (weight) independent set in G

NP-Hard, and also very hard to approximate!

No $\Omega\left(\frac{1}{n^{1-o(1)}}\right)$ approx. unless P = NP [Hastad99, Zuckerman06]

Special Cases and Applications

- Sparse graphs: bounded degree or degeneracy
- Interval graphs, chordal graphs, …
- Geometric intersection graphs
- Inductively k-independent graphs
- Perfectly k-orientable graphs
- …

Theory but also because of applications

Elimination Graphs

- **Inductively k-independent graphs**

$\subset$

- **Perfectly k-orientable graphs**

Capture several classes in *parameterized* and *unified* way, both sparse and dense

MWIS has $\frac{1}{k}$ and $\frac{1}{2k}$ approximation in above classes

[AADK02] [KT14]

Elimination Graphs

- **Inductively k-independent graphs**
- **Perfectly k-orientable graphs**

MWIS has $\frac{1}{k}$ and $\frac{1}{2k}$ approximation in above classes

This talk: obtain results when objective is *submodular*

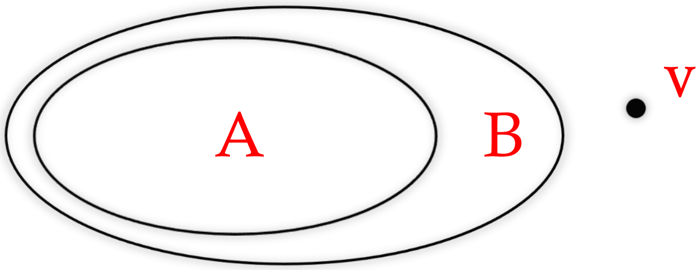
Submodular Set Functions

Real-valued set function $f: 2^V \to R$ is **submodular** if

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \quad \forall A, B$$

Equivalently:

$$f(A+v) - f(A) \geq f(B+v) - f(B) \quad A \subset B, v \notin \mathrm{B}$$



Submodular Set Functions

Real-valued set function $f: 2^V \rightarrow R$ is **submodular** if

$f(A+v) - f(A) \geq f(B+v) - f(B) \quad A \subset B, v \notin \mathrm{B}$

f is *monotone* if $f(A) \leq f(B)\ for\ all\ A \subseteq B$

f is *non-negative* if $0 \leq f(A)\ for\ all\ A$

Assume $f(\emptyset) = 0$

Motivation

Many positive results and applications on *constrained* submodular set function maximization

max f(S) where S is *independent*

- matroid and matroid like constraints
- knapsack constraints and packing integer programs
- …

Extend positive results to graph independent sets

Max Submod Independent Set

$G = (V, E)$ undirected graph

$S \subseteq V$ is an independent/stable set if G[S] has no edges

Problem: given G and submodular $f: 2^V \to R_+$

max f(S) where S is independent set in G

Result 1

- **k-perfectly orientable graphs:** *randomized* algorithm that yields $\frac{1}{e(k+1)}$ approx. for non-neg submod funcs. Slightly better for monotone functions (see paper)
  - Based on multilinear relaxation and rounding approach
  - Randomized and inefficient
  - But can be parallelized

Result 2

- **Inductively k-independent graphs:** *deterministic* algorithm that yields $\frac{1}{k+1+2\sqrt{k}}$ approx. for monotone functions and *randomized* $\Omega\left(\frac{1}{k}\right)$ approx. for non-monotone functions
  - Based on primal-dual technique
  - Very simple and efficient
  - First deterministic approx. for several special cases such as interval graphs, pseudo-disk graphs etc

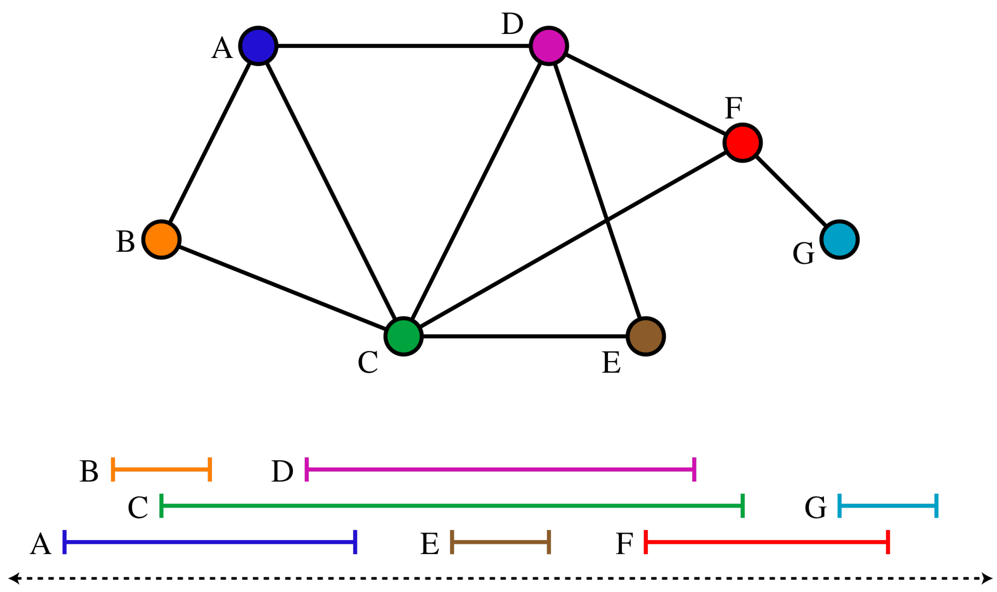
Inspiration and Take Aways

Results are based on combining existing ideas but …

- Graph classes are *unifying* and *interesting*
- *Primal-dual* for submod max was first done by [Levin-Wajc21] for streaming application. This paper shows applications for offline
  - For interval graphs first deterministic approx. and ratio of ¼ matches previous best randomized [Feldman13]
- Connections to streaming algorithms
- Several open problems (see paper)

Interval Graphs

- Given n intervals $I_1, I_2, \ldots, I_n$ where $I_i = [a_i, b_i]$
- Defines *intersection graph*: vertices correspond to intervals and two intervals connected iff they overlap



src: Wikipedia

Interval Graphs

Max weight independent set in interval graphs is easy.
Several ways to see this but take graph theoretic view

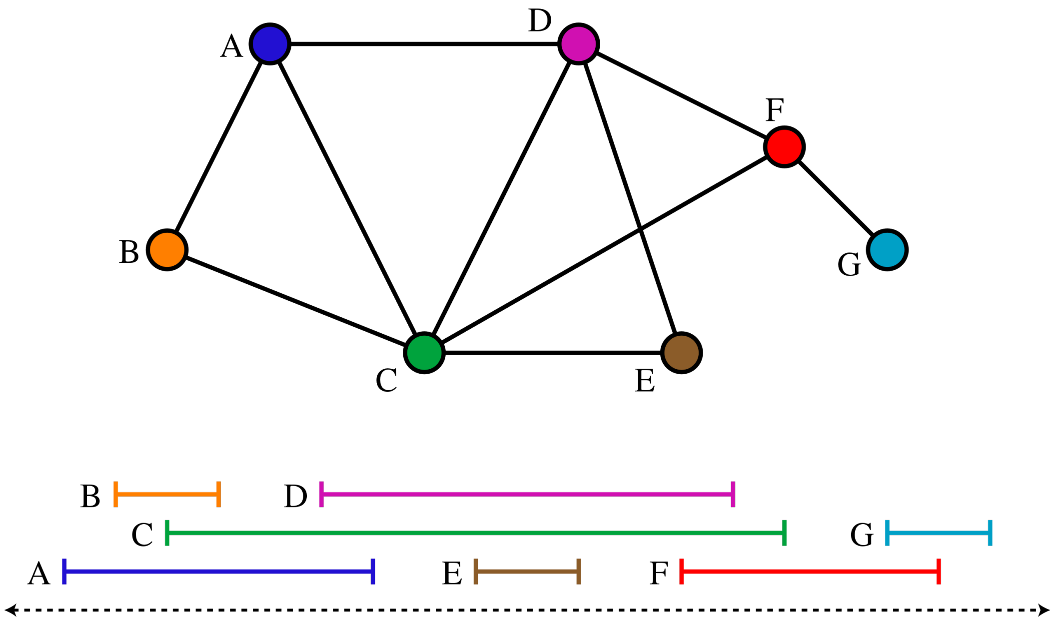
They are *chordal* and have *perfect elimination ordering*

Vertices can be ordered as $v_1, v_2, \dots, v_n$ such that for each i, $N(v_i) \cap \{v_i, v_{i+1}, \dots, v_n\}$ is a *clique*

$N(v_i)$ is set of neighbors of v_i

Sort intervals by right end point

B A E D C F G



Inductively k-independent graphs

[AADK02, YB12]

G is inductively k-independent if there is a vertex ordering $v_1, v_2, \ldots, v_n$ such that for each i, *max independent set of* $G[N(v_i) \cap \{v_i, v_{i+1}, \ldots, v_n\}] \leq k$

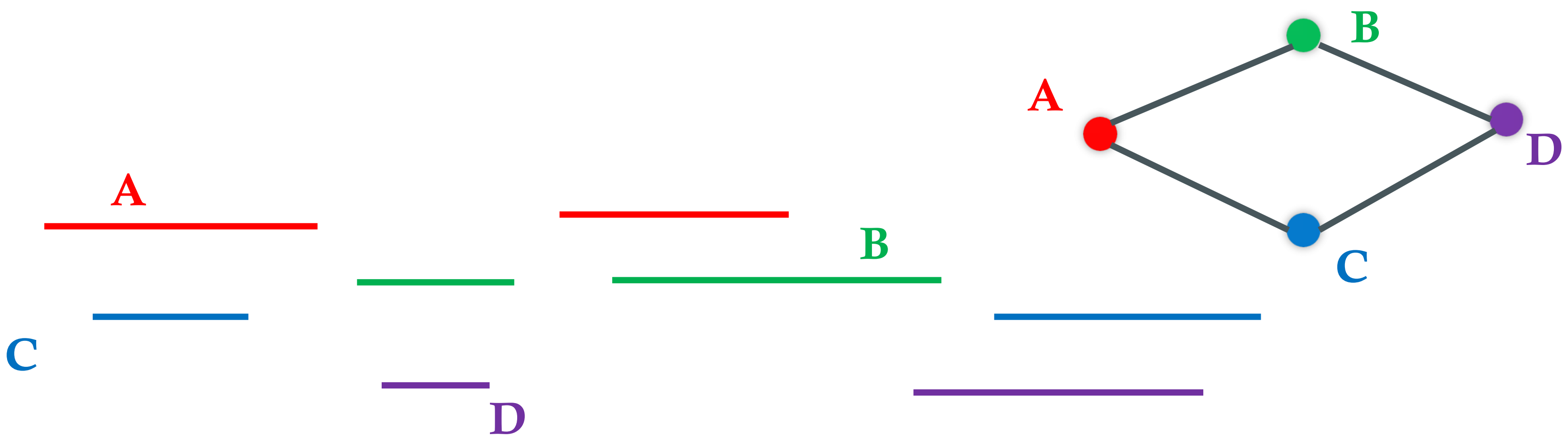
Easier condition: $G[N(v_i) \cap \{v_i, v_{i+1}, \ldots, v_n\}]$ can be covered by k cliques

Chordal/interval graphs: k =1 due to single clique

Matchings in graphs: line-graph has k=2 since for any edge its neighbors are covered by two cliques (any ordering will work!)

t-interval graphs

n objects: object is a collection of t disjoint intervals, form intersection graph



Max-Indep-Set hard and non-obvious to approximate even when t = 2 [BNSS06]

k-perfectly-orientable graphs

[KT14]

G=(V,E) has an *orientation* H=(V,A) such that for each vertex v, $N_H^+(v)$ can be covered by k cliques

$N_H^+(v)$ is out-neighborhood of v

Orientation is powerful. Exist 2-perfectly orientable graphs that are *not* inductively $\sqrt{n}$ –independent

t-interval graphs are 2t-perfectly-orientable

LP Relaxation

Assume ordering $v_1, v_2, \dots, v_n$ is given

Variable x_i whether v_i in independent set

Constraints are simple:

$$x_i + \sum_{j \in A_i} x_j \le k \quad for\ all\ i$$
$$x_i \in [0,1]\ for\ all\ i$$

A_i: $N(v_i) \cap \{v_i, v_{i+1}, \dots, v_n\}$ *or* $N_H^+(v_i)$

Multilinear Relaxation Based Algorithms

max F(x) such that

$$x_i + \sum_{j \in A_i} x_j \leq k \quad for\ all\ i$$
$$x_i \in [0,1]\ for\ all\ i$$

A_i: $N_H^+(v_i)$

Can obtain (1-1/e) approx. for relaxation via continuous greedy

Round x via contention resolution scheme

Randomized Rounding

Round x via contention resolution scheme

Very simple and natural here (implicit in [Feldman13] for interval graphs)

- Pick each v_i with probability $\frac{cx_i}{k}$ for some constant c. Let R be random set
- Discard v_i from R if R has *any* vertex from neighborhood $N_H^+(v_i)$
- Output altered solution R'

(balance parameters appropriately to get best ratio)

Primal-Dual

Inspired by [Levin-Wajc'21]

Only for inductively k-independent

Want LP relaxation so use "concave extension" $f^+(x)$

max $f^+(x)$ such that

$$x_i + \sum_{j \in A_i} x_j \leq k \quad for\ all\ i$$
$$x_i \in [0,1]\ for\ all\ i$$

Primal-Dual

$f^+(x)$ hard to evaluate but won't need to solve since we we are using primal-dual

max $f^+(x)$ such that

$$x_i + \sum_{j \in A_i} x_j \leq k \quad for\ all\ i$$
$$x_i \in [0,1]\ for\ all\ i$$

Rewrite above as big LP from definition of $f^+(x)$

Primal and Dual

$$\max \sum_{S\subseteq V} \alpha_L f(L)$$

$$\begin{aligned}
\sum_{L\subseteq V} \alpha_L &= 1 \\
\sum_{L\ni v_i} \alpha_L &= x_i \quad i \in [n] \\
x_i + \sum_{v_j \in A_i} x_j &\leq k \quad i \in [n] \\
x_i &\geq 0 \quad i \in [n]
\end{aligned}$$

$$\min \mu + k \sum_{i=1}^{n} y_i$$

$$\begin{aligned}
\mu + \sum_{v_i \in L} z_i &\geq f(L) \quad L \subseteq V \\
y_i + \sum_{v_j \in B_i} y_j &\geq z_i \quad i \in [n] \\
y_i &\geq 0 \quad i \in [n]
\end{aligned}$$

Primal-Dual Algorithm

Related to local-ratio and stack-based algorithms for MWIS (see paper for refs). Restrict attention to *monotone* functions. Non-neg requires more work.

Two phases

- **Phase 1:** process vertices in order $v_1, v_2, \ldots, v_n$ and choose a subset S stored in a *stack*
- **Phase 2:** Process S in *reverse* order by popping. Obtain a subset S' that is an independent set

Phase 1

- Initialize stack S to empty
- For i = 1 to n do
  - Process v_i
  - $C_i = S \cap N(v_i)$ *//the vertices in S that conflict with* v_i
  - If $f_S(v_i) \geq (1+\beta) \sum_{v_j \in C_i} w_j$ then
    - // v_i is significantly better than conflict set so add
    - $S.push(v_i)$
    - $w_i = f_S(v_i)$ // w_i is value when added

Phase 2

- Initialize S' to empty
- While S is not empty
  - $v = S.pop$
  - If $S' + \mathrm{v}$ is an independent set then
    - $S' \leftarrow S' + \mathrm{v}$
- Output S'

Open Problem

¼ approx. for interval graphs when f is monotone

- Via primal-dual
- And via multilinear relaxation

Can we do better?

Thank You!

Backup Slides

Interval Graphs

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