Maximizing A Submodular Set Function (Revisited)

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Motivation

[Nemhauser-Wolsey-Fisher 78]

An analysis of the approximations for maximizing submodular set functions I, Math Programming, 1978

[Fisher-Nemhauser-Wolsey 78]

An analysis of the approximations for maximizing submodular set functions II, Math Programming Study, 1978

Interesting applications

[CK'00,FHR03,CK'04,MV'04,CP'05,FGMS'06,...]

Outline

- Problem Definition
- Greedy algorithm’s performance
- Example application: generalized assignment
- LP Relaxation to improve upon Greedy

Submodular set function

E : discrete ground set

f: $2^E \rightarrow \mathcal{R}^+$

f is submodular iff

$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A,B \subseteq V$

equivalently

$f(A+\{e\}) - f(A) \leq f(B+\{e\}) - f(B)$ if $B \subset A$

Monotone Submodular set function

E : discrete ground set

f: $2^E \rightarrow \mathcal{R}^+$

f is submodular iff

$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A,B \subseteq V$

monotone (non-decreasing)

$f(A) \geq f(B)$ if $B \subseteq A$

$f(\emptyset) = 0$

Max Submodular Set Function

Discrete set E

Submodular set function f: $2^E \to \mathcal{R}^+$

$\max_{S \subseteq V}$ f(S)

s.t

S $\in$ *allowed sets*

f given by a valuation *oracle*

allowed sets also specified as an *oracle*

Max Submodular Set Function

Discrete set E

Submodular set function f: $2^E \rightarrow \mathcal{R}^+$

$$\max_{S \subseteq V} f(S)$$

s.t

$S \in$ *independent sets* $(\mathcal{I} \subseteq 2^E)$

f given by a valuation *oracle*

independent sets also specified as an *oracle*

Greedy Algorithm

start with $S = \emptyset$

repeat

  $A = \{ e \mid S+e$ is independent $\}$

  if $A = \emptyset$ then STOP and output S

  $e' = \text{argmax}_{e \in A} f(S+e) - f(S)$

  $S = S+e'$

When is Greedy optimal?

[Edmonds 6?]

f is *modular* (linear): $f(S) = \sum_{e \in S} w(e)$

max $f(S)$

s.t

S is an *independent* set in a matroid M on E

Example: max/min weight spanning tree (Kruskal's algorithm)

Matroid

discrete set E, $\mathcal{I} \subseteq 2^E$

$M = (E, \mathcal{I})$ is a matroid if

- $\emptyset \in \mathcal{I}$
- $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$
- $A, B \in \mathcal{I}$ and $|A| < |B|$ implies, there exists $e \in B \setminus A$ s.t $A + e \in \mathcal{I}$

$A \in \mathcal{I}$ is "independent"

A is a *cycle* if it is minimally non-independent

(there is e s.t $A - e$ is independent)

Examples of matroids

- $\mathcal{I} = \{ A \mid |A| \leq k \}$
- $E_1, E_2, \ldots, E_h$ a *partition* of E, $k_1, \ldots, k_h \in \mathcal{Z}^+$
  $\mathcal{I} = \{ A \mid |A \cap E_i| \leq k_i \}$
- Laminar families (trees) with integer capacities
- Graphic matroid. $G = (V, E)$. A is independent iff A induces a forest
- E is a collection of vectors in a vector space, A is independent if the vectors are linearly indep

When is Greedy optimal?

[Edmonds 6?]

f is linear

max f(S)

s.t

S is an *independent* set in a matroid M on E

Greedy for p-independence families

[Jenkyns 76, Korte-Hausman 78]

If f is linear Greedy is a $1/p$ approximation alg. for p-independence families

Example: Greedy for maximum weight matching in a graph is a ½ approximation

p-independence families

Discrete Set E, Independence family $\mathcal{F} \subseteq 2^E$

Downward closed: $A \in \mathcal{F}$ and $B \subset A \Rightarrow B \in \mathcal{F}$

$A \in \mathcal{F}$ is "independent"

A is a *circuit* or *minimally non-independent* if there is e s.t $A - \{e\}$ is independent

p-independent family: for every $A \in \mathcal{F}$, $A+\{e\}$ has at most p circuits

or

there exists $B = \{e_1, e_2, ..., e_p\}$ s.t $A - B + e \in \mathcal{F}$

Examples: p-independence families

$p = 2$

Intersection of 2 matroids

Example: $G = (V_1, V_2, E)$ is a *bipartite* graph

$A \subseteq E$ is independent if A induces a *matching* in G

(not matroid intersection)

$G = (V, E)$ is a graph. $A \subseteq E$ is independent if A induces a *matching* in G

Greedy and implicit set systems

Often E is *implicit* and $|E|$ is *exponential* in input problem size

Greedy can be implemented if the step

$e' = \text{argmax}_{e \in A} f(S+e) - f(S)$

can be implemented in *polynomial* time

$\alpha \leq 1$ approx for oracle $\Rightarrow$ Greedy is α/p approx for linear f

On to submodularity

Examples of submodular set functions

- set systems
- from matroids (rank functions etc)
- cut functions in graphs (not monotone)
- ...

Submodularity from Set Systems

$X_1, X_2, \ldots, X_n$ subsets of $\mathcal{U}$

$E = \{1, 2, \ldots, n\}$

for $A \subseteq E$, let $X(A) = \cup_{i \in A} X_i$ (elements covered by A)

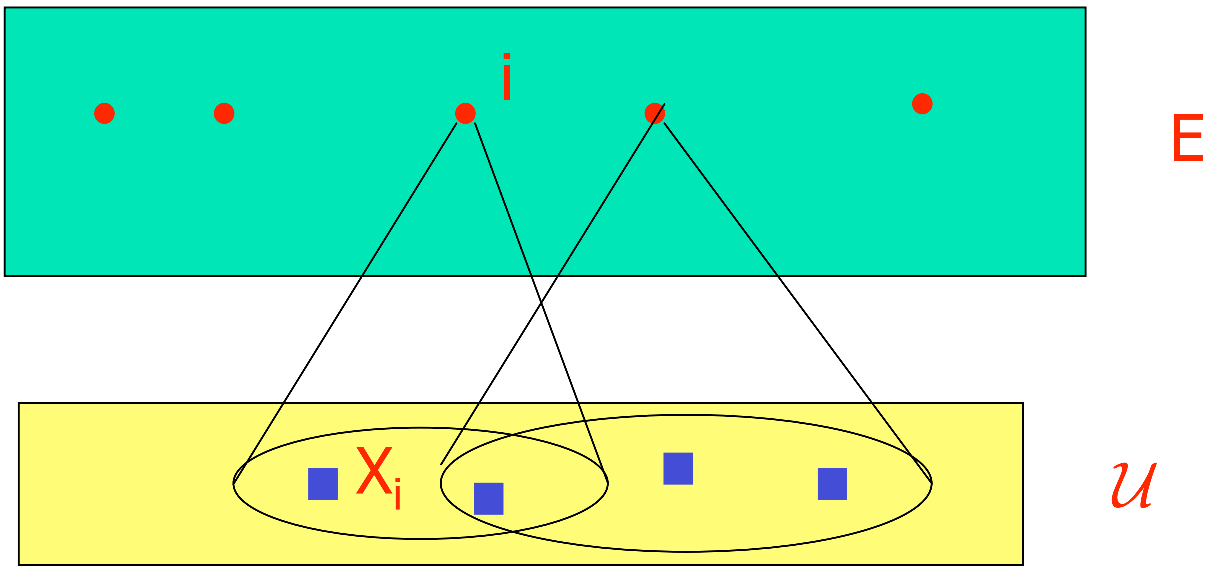
for $A \subseteq E$, $e \in \mathcal{U}$, $c(A, e) = \{i \in A \mid e \in X_i\}$

Coverage: $f(A) = | X(A) |$ = Multi-coverage: $f(A) = \sum_{e \in X(A)} \max\{k(e), |c(A,e)|\}$

Weighted coverage: $f(A) = \sum_{e \in X(A)} w(e)$

Weighted max coverage: $f(A) = \sum_{e \in X(A)} \max_{i \in c(A,e)} w(e,i)$

Set Systems



Maximizing Submodular Set Functions

max $f(S)$

s.t

$|S| \leq k$

NP-hard (for example via Maximum Coverage)

How good is Greedy?

max $f(S)$

s.t

$|S| \leq k$

Greedy yields a $1-1/e$ approximation

[Nemhauser-Wolsey-Fisher 78]

Special case: Maximum Coverage

$X_1, X_2, \ldots, X_n$ subsets of ground set $\mathcal{U}$

Pick k out of n sets to maximize size of their union

$E = \{1, 2, \ldots, n\}$

$f(A \subset E) = | \cup_{i \in A} X_i |$

max $f(S)$

$|S| \leq k$

Maximum Coverage

$X_1, X_2, \ldots, X_n$ subsets of ground set $\mathcal{U}$

Pick k out of n sets to maximize size of their union

$E = \{1, 2, \ldots, n\}$

$f(A \subset E) = |\cup_{i \in A} X_i|$

Unless $NP \subset DTIME(n^{\log \log n})$, there is no $1-1/e - \varepsilon$ approximation for Maximum Coverage [Feige98]

Greedy for p-independence families

max f(S)

s.t

S is independent in $\mathcal{I}$

[Fisher-Nemhauser-Wolsey 78]

Greedy is a 1/(p+1) approximation

½ when (E, $\mathcal{I}$) is a matroid

(α appox for oracle, Greedy is $\alpha/(p+\alpha)$ approx)

Other results in [NWF78, FNW78]

- Simpler Greedy for interesting special class
- 1/2 approx for local search algorithm for $p=1$
- bad performance for local search when $p > 1$
- LP formulation/analysis for some special cases
- Examples

Example: Multiple Knapsack and Generalized Assignment

- Multiple knapsack (uniform):
  - m knapsacks of size B each.
  - n items. item j has size s_j, profit p_j.
- Multiple knapsack (non-uniform):
  - knapsack sizes $B_1 < B_2 < \ldots < B_m$
- Genarilized assignment:
  - item j has size s_{ij} and profit p_{ij} to go into knapsack i
  - knapsack sizes 1 (wlog)

Greedy for Multiple Knapsack

Use single knapsack FPTAS iteratively

How good is it?

Greedy for Multiple Knapsack

Use single knapsack FPTAS iteratively

How good is it?

[C-Khanna '00]

Greedy is $1-1/e - \varepsilon$ approx for uniform

(max $f(S)$ s.t $|S| \leq m$)

Greedy is $1/2 - \varepsilon$ approx for non-uniform

(max $f(S)$ s.t S is independent in a matroid)

Generalized Assignment (GAP)

m knapsacks

Each item j has per knapsack size and profit

s_{ij} : size of item j in knapsack i

p_{ij}: profit of packing item$j$ in knapsack i

Pack items into bins to maximize profit

X - set of items

GAP as Max f(S)

$E_i = \{ A \subseteq X \mid s_i(A) \leq 1 \}$

$A \in E_i$ implies items in A fit into knapsack i

$E = \uplus_{i=1}^{m} E_i$

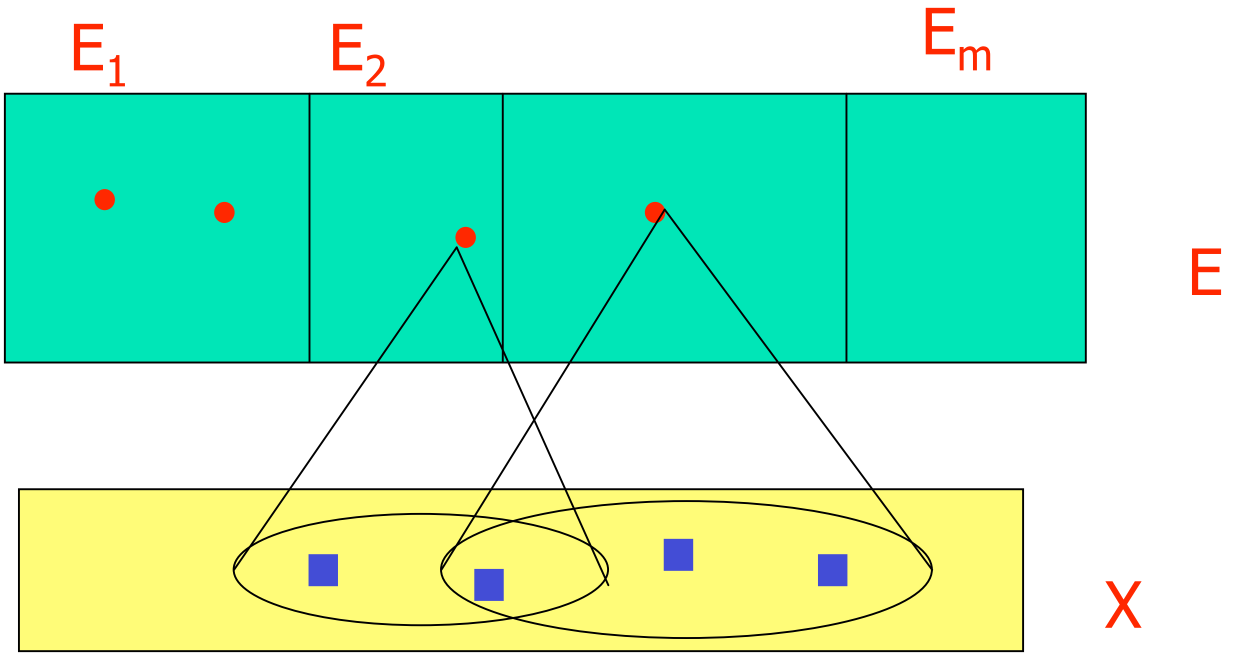
Given $A \subseteq E$,

$f(A) = \sum_{j \in X} \max_{i:\, S \in E \cap E_i,\, j \in S} p_{ij}$

GAP: max $f(S)$ s.t

$|S \cap E_i| \leq 1$ (simple partition matroid constraint)

Set system for GAP



Greedy for Generalized Assignment

p_j = current accrued profit of item j

$p_j = 0$ for all i (all items have initial profit 0)

For $i = 1$ to m do

$p'_{ij} = \max\{0, p_{ij} - p_j\}$ (residual profit of item j)

Solve knapsack for bin i using p'_{ij} s as profits (all items)

Let S_i be items assigned in knapsack

If j is in S_i, $p_j = p_j + p'_{ij}$

Remark: an item can be packed multiple times!

Greedy for Gen Assignment

Greedy is a $\frac{1}{2} - \varepsilon$ approximation!

Local search also $1/2 - \varepsilon$ approximation!

LP based 1/2 approx [C-Khanna'00] (using [Shmoys-Tardos'91])

[Fleischer-Goemans-Mirrokni-Sviridenko'06]

Improved $1-1/e-\varepsilon$ using stronger LP

[Feige-Vondrak'06]

$1-1/e+\delta$ using same LP

Other applications

Greedy + additional ideas

k-repairman

Orienteering (with Time Windows)

Group Steiner

...

[C-Kumar 04, C-Pal 05]

Is Greedy the best one can do?

Greedy gives $1/(p+1)$ approximation

For large p, unless *P = NP*, no algorithm better than $\Omega(\log p/p)$ [Hazan-Safra-Schwartz '03]

(max p-dimensional matching, f is modular)

(Single) Matroid constraint

Given E, f and $M=(E, \mathcal{I})$

$$\max_{S \subseteq V} f(S)$$

s.t

$$S \in \mathcal{I}$$

Question: can we do better than Greedy (1/2)?

(1-1/e) is best possible even for simple $|S| \leq k$

(Single) Matroid constraint

Given E, f and $M=(E, \mathcal{I})$

$$\max_{S \subseteq V} f(S)$$

s.t

$$S \in \mathcal{I}$$

Theorem: 1-1/e for some restricted f such as f arising from set systems.

(Single) Matroid constraint

Theorem: 1-1/e for some restricted f such as f arising from set systems.

With some work, 1-1/e for generalized assignment (known from [Fleisher-Goemans-Mirrokni-Sviridenko'06] essentially a reinterpretation)

(Single) Matroid constraint

Theorem: can obtain 1-1/e for some restricted f such as f arising from set systems.

Proof idea: use LP relaxation and rounding

(inspired by pipage-rounding [Ageev-Sviridenko 04])

(Single) Matroid constraint

What does f need to satisfy?

$f(y)$ defined for $y \in \{0,1\}^n$

Extend f to $[0,1]^n$ (relaxation to fractional values)

For each $y \in [0,1]^n$ if $E_y[f(y)] \geq \alpha\ f(y)$, then α approximation

(easy for set system type problems)

Useful settings

Maximum coverage under following constraints

- $|S| \leq k$ (Greedy gives $1-1/e$)
- Sets are colored and can pick at most k_i colors from sets of color i (partition matroid)
- Color class constraints + total # $\leq k$ (laminar family constraint)
- Sets are associated with edges of a graph and picked sets induce a tree/forest

LP Relaxation

max $f(S)$

$S \in \mathcal{I}$

$y_i \in \{0,1\}$ to indicate if $i \in E$ is in solution

max $f(y)$

$y \in P(M)$

$y_i \in \{0,1\}$

$P(M)$: polytope of matroid M

LP Relaxation

max $f(y)$

$y(A) \leq r(A) \quad S \subseteq E$

$y_i \in [0,1] \quad i \in E$

Need to extend f from $\{0,1\}^n$ to $[0,1]^n$

Easy/natural for coverage type problems

LP Relaxation

Need to extend f from $\{0,1\}^n$ to $[0,1]^n$

$f(y) = \sum_{e \in \mathcal{U}} \max \{1, \sum_{e \in X_i} y_i \}$

What about a general f given as oracle?

[FNW'78] formulation if f is *matroid type*

Observation: integer valued f is of matroid type using [Helgason'74] (weakly polynomial)

(useful in other contexts?)

LP Rounding

If $y_i \in \{0, 1\}$, reduce problem to smaller instance

Can assume y is completely fractional

set A is tight if $y(A) = r(A)$

A, B tight $\Rightarrow$ $A \cup B$, $A \cap B$ also tight

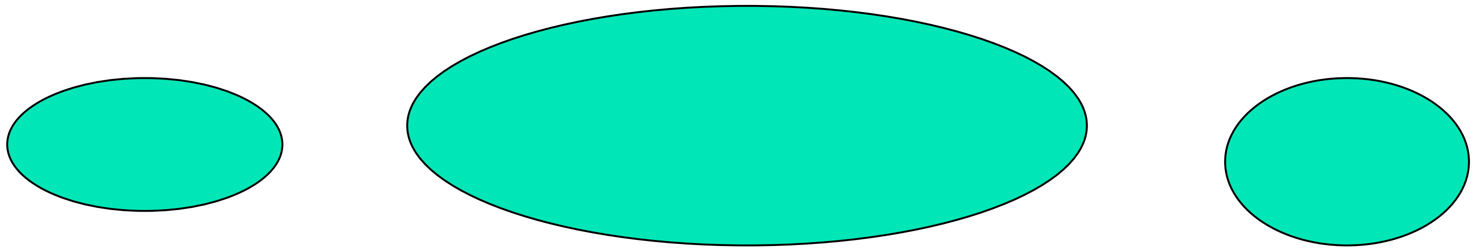
$\Rightarrow$ *minimal tight sets are disjoint*

LP Rounding

set A is tight if $y(A) = r(A)$

A, B tight $\Rightarrow$ $A \cup B$, $A \cap B$ also tight

$\Rightarrow$ *minimal tight sets are disjoint*

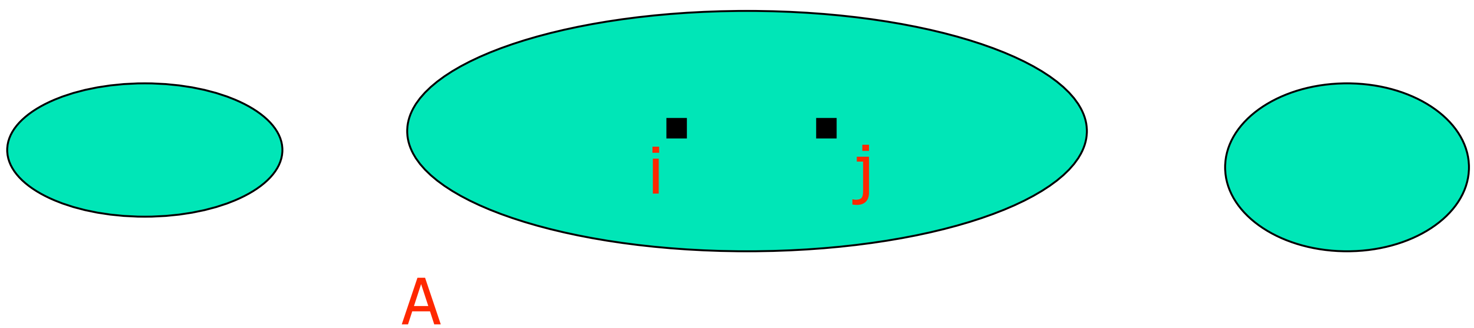


LP Rounding

set A is tight if $y(A) = r(A)$

A, B tight $\Rightarrow$ $A \cup B$, $A \cap B$ also tight

$\Rightarrow$ *minimal tight sets are disjoint*



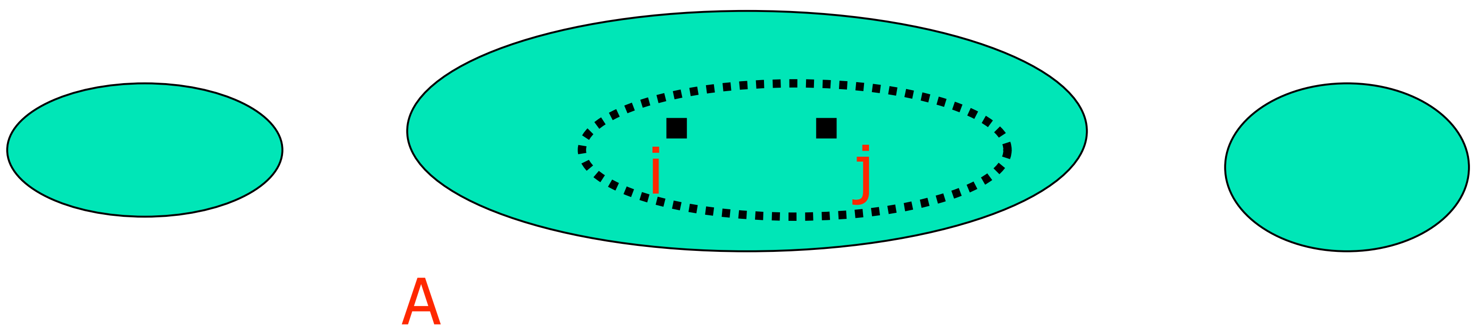
$y_i = y_i + \varepsilon$, $y_j = y_j - \varepsilon$

LP Rounding

set A is tight if y(A) = r(A)

A, B tight ⇒ A∪ B, A ∩ B also tight

⇒ *minimal tight sets are disjoint*



Claim: New minimal tight set *strictly contained* in A

LP Rounding

Choose $\varepsilon > 0$ or < 0?

According to potential function E[f(y)], which ever is greater

Lemma: E[f(y)] is convex in ε for any two variables y_i, y_j if f is submodular

Conclusions

Remember 1/(p+1) approx ratio of Greedy

submodularity + independence constraints quite general

LP relaxation yields 1-1/e for one matroid constraint for some f – better than ½ given by Greedy (*need to solve LP, not always possible for exponential sized systems*)

Open Problem: can we obtain 1-1/e for *all* submodular f ? (exists an LP relaxation for integer valued f using [Helgason 74])

Thank You!

Simple question

Given graph $G = (V, E)$

$y(e) \in [0,1]$ for $e \in E$

y belongs to the spanning tree polytope

(that is y is a convex combination of spanning trees)

Randomly choose each e with prob $y(e)$

What is the expected number of connected components in resulting graph?

Conjecture: $\leq n/e$ *(known ?)*

Integer valued f - polymatroids

Polymatroids have nice polyhedral description

[Edmonds]

(useful for LP formulations)

[Helgason74]

Every polymatroid $f: 2^V \rightarrow \mathcal{Z}^+$ has an underlying matroid $M = (V', \mathcal{I})$ s.t f is the rank function of M

$|V'| = \sum_{i \in V} f(i)$ (weakly polynomial)

(Single) Matroid constraint

What does f need to satisfy?

$f(y)$ defined for $y \in \{0,1\}^n$

Extend f to $[0,1]^n$ (relaxation to fractional values)

For each $y \in [0,1]^n$ if $E_y[f(y)] \geq \alpha \; f(y)$, then α approximation

Rounding in Matroids

set A is tight if $y(A) = r(A)$

Pick *minimal* tight set H and let y_i, y_j be fractional in H
For some small $\varepsilon > 0$,

- $y_i = y_i + \varepsilon$, $y_j = y_j - \varepsilon$
- $y_i = y_i - \varepsilon$, $y_j = y_j + \varepsilon$

are both feasible
Choose max ε and do one of above

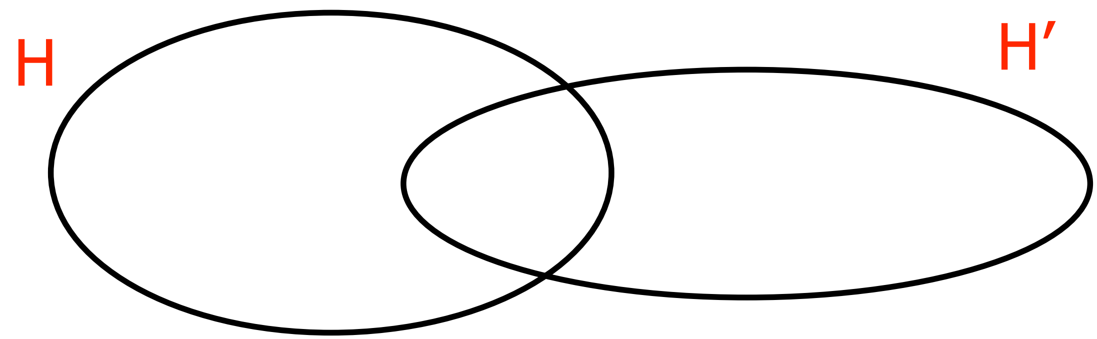
Claim: new minimal tight set will be contained in H
Process will converge – eventually some $y_i = 1$ or 0

Rounding in Matroids

Claim: new minimal tight set will be contained in H

A, B tight implies A ∩ B and A ∪ B are tight

Minimal tight sets do not interesect



Rounding contd..

Choose which option?

[Ageev-Sviridenko] pipage rounding

Define potential function $F(y)$ with following properties

- $F(y) = f(y)$ for integer y
- $F(y) \geq \alpha f(y)$ for all y
- $g_{ij}(\varepsilon, y)$ is convex in ε when changing of two variables y_i and y_j

Rounding contd..

Choose which option?

[Ageev-Sviridenko] pipage rounding

Define potential function $F(y)$ with following properties

- $F(y) = f(y)$ for integer y
- $F(y) \geq \alpha f(y)$ for all y
- $g_{ij}(\varepsilon, y)$ is convex in ε when changing of two variables y_i and y_j

Rounding contd ...

y_1 is obtained from y by $y_i + \varepsilon$, $y_j - \varepsilon$

y_2 is obtained from y by $y_i - \varepsilon$, $y_j + \varepsilon$

By convexity condition,

$F(y_1) \geq F(y)$ *or* $F(y_2) \geq F(y)$

Hence after rounding

$f(y') = F(y') \geq F(y) \geq \alpha\, f(y) \geq \alpha\, OPT$

Potential Function?

How do we get a potential function?

[Ageev-Sviridenko] simple explicit potential function for maximum coverage

$\sum_e (1 - \Pi_{e \in X_i} y_i)$

Natural potential function: $F(y) = E_y[f(y)]$

independently set i to 1 with probability y_i

Potential Function?

Natural potential function: $F(y) = E_y[f(y)]$

Lemma: $g_{ij}(\varepsilon, y)$ is convex if f is submodular!

Only thing to do: prove $F(y) \geq \alpha f(y)$

For many set system type problems $\alpha = 1\text{-}1/e$
(easy to prove)