Approximation Algorithms for Network Design in Non-Uniform Fault Models

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Network Design for Connectivity

Given graph/network $G=(V,E)$ find a *cheap* subgraph H such that H satisfies some *connectivity* requirement

This talk:

- Cost is on edges, $c(e)$ for each edge e and $cost(H)$ is total cost of edges in H
- Undirected graphs
- Edge connectivity

Network Design for Connectivity

Given graph/network G=(V,E) find a *cheap* subgraph H such that H satisfies some *connectivity* requirement

Examples:

- Minimum spanning tree (MST)
- Shortest s-t path
- Steiner tree and Steiner forest (NP-Hard, APX-Hard)
- Metric-TSP, ATSP (NP-Hard, APX-Hard)

Network Design for Connectivity

Given graph/network G=(V,E) find a *cheap* subgraph H such that H satisfies some *connectivity* requirement

Examples: *higher connectivity*

- Min-cost k-edge-disjoint s-t paths
- k-ECSS min-cost k-edge-connected subgraph (NP-Hard for k=2)
- Survivable Network Design

Survivable Network Design Problem (SNDP)

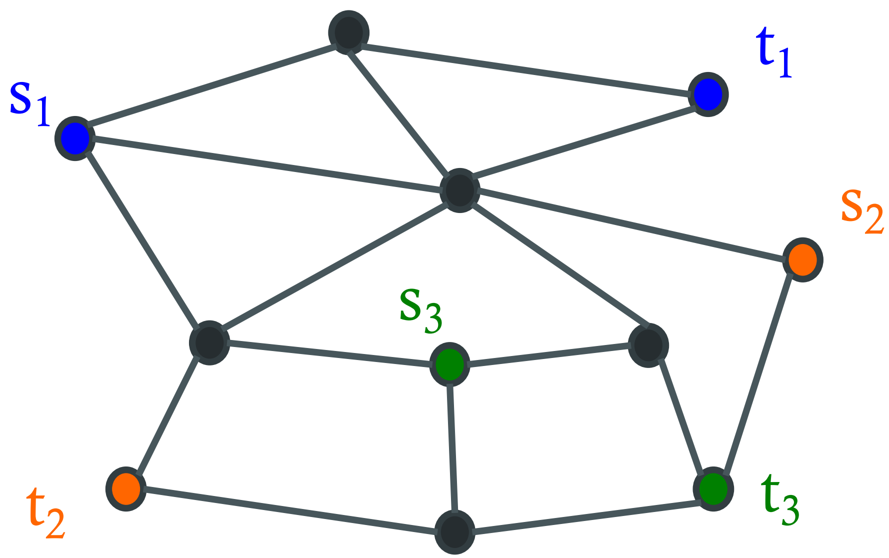
Input:

- undirected graph G=(V,E)
- integer requirement r(st) for each pair of nodes st

Goal: *min-cost* subgraph H of G s.t H contains r(st) *edge-disjoint* paths for each pair st

One motivation: *fault tolerant* network design

$r(s_1 t_1) = 2$
$r(s_2 t_2) = 2$
$r(s_3 t_3) = 1$



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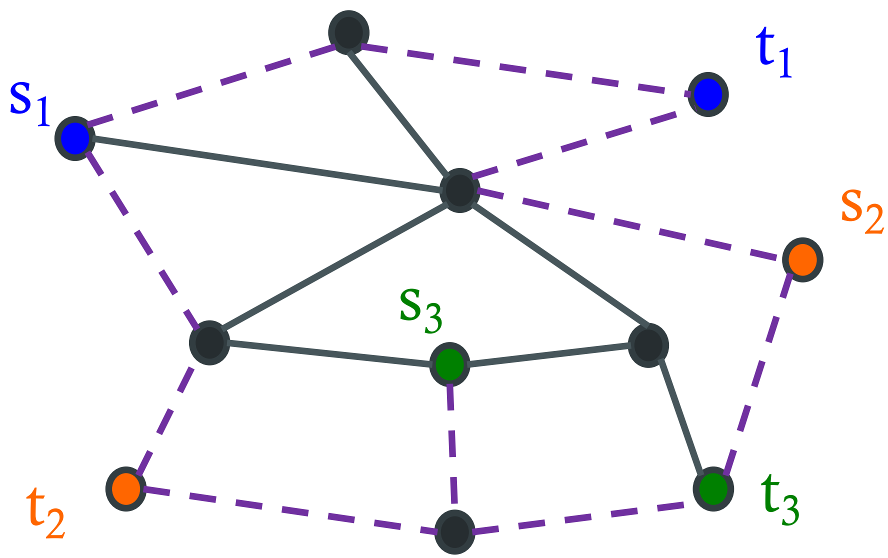

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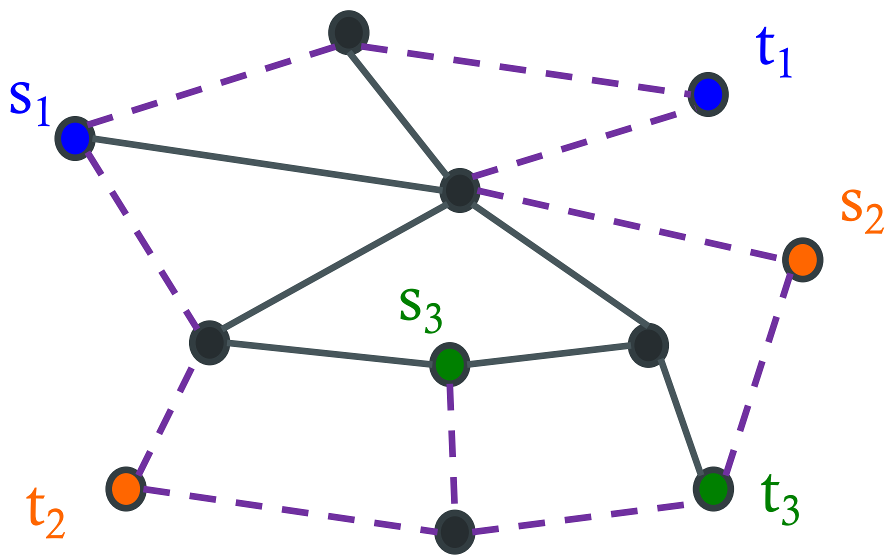


Theorem: [Jain'98] 2-approximation via *iterated rounding*

Theorem: [WGMV'95] $2r_{max}$ -approximation via primal-dual+augmentation

Non-Uniform Fault Models

k-edge-connectivity: robustness to *any* k-1 edge failures

Non-uniform models

- Edges/links vary in their failure profiles
- Correlations between edge failures

How to model?

This Talk

Two models

- Flexible Graph Connectivity [Adjiashvili 2013, Adjiashvili-Hommelshem-Muhlenthaler 2020]
- Bulk Robust [Adjiashvili-Stiller-Zenklusen 2015]

Flexible Graph Connectivity

[Adjiashvili'13]

G=(V,E) representing underlying network

- E partitioned into *safe edges* S and *unsafe edges* U
- Only *unsafe edges* can *fail*
- Design network robust to failures
- Initial model considered single pair (s,t)

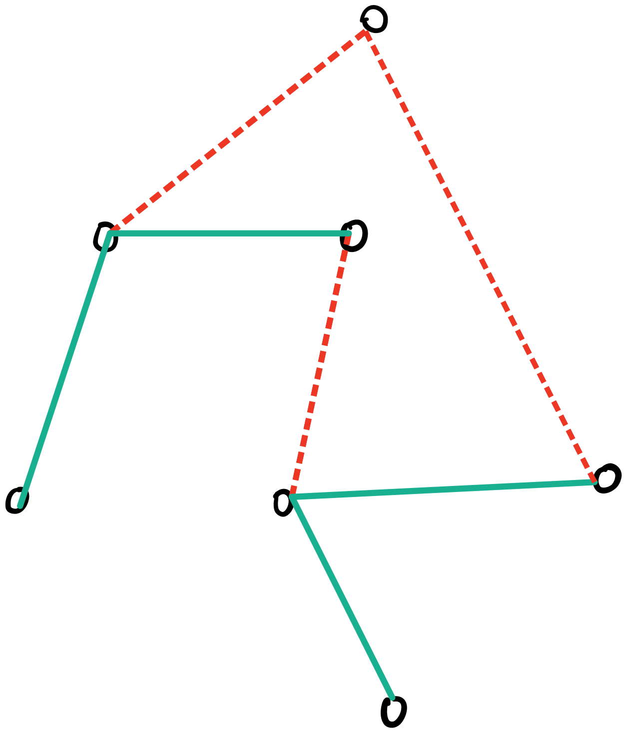
Flexible Graph Connectivity

[Adjiashvili-Hommelshem-Muhlenthaler 2020]

G=(V,E) representing underlying network

- E partitioned into *safe edges* S and *unsafe edges* U
- Only *unsafe edges* can *fail*

Find min-cost subgraph $H=(V, E_H)$ such that H is connected under any *one unsafe edge failure*



Flexible Graph Connectivity

- E partitioned into *safe edges* S and *unsafe edges* U
- Only *unsafe edges* can *fail*

Find min-cost subgraph $H=(V, E_H)$ such that H is connected under any *one unsafe edge failure*

Questions:

- What is the problem if all edges are safe?
- If all edges are unsafe?

Flexible Graph Connectivity

- E partitioned into *safe edges* S and *unsafe edges* U
- Only *unsafe edges* can *fail*

Find min-cost subgraph $H=(V, E_H)$ such that H is connected under any *one unsafe edge failure*

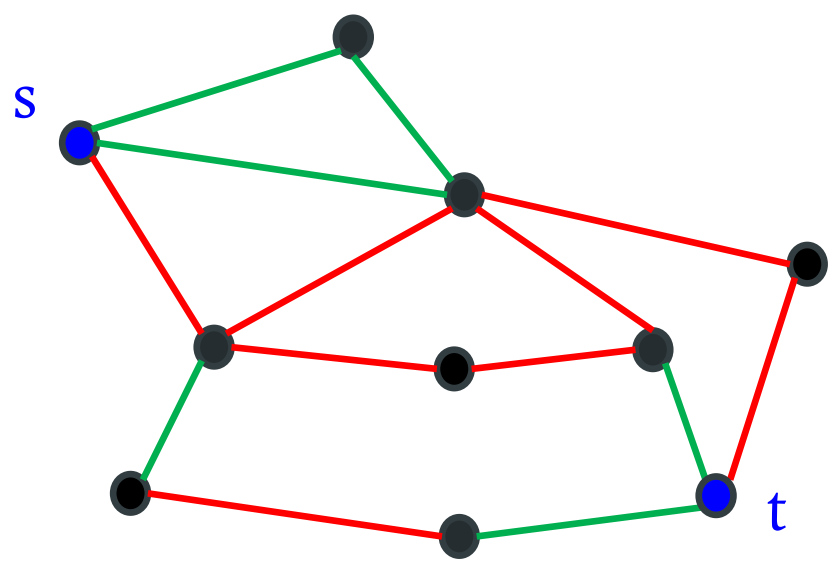
Questions:

- What is the problem if all edges are safe? MST
- If all edges are unsafe? 2-ECSS

Flexible Connectivity Model

E partitioned into *safe edges* S and *unsafe edges* U.

Definition: A pair of vertices (s,t) are (p,q)-flex-connected if s and t are p-edge-connected after removing *any set* of q unsafe edges



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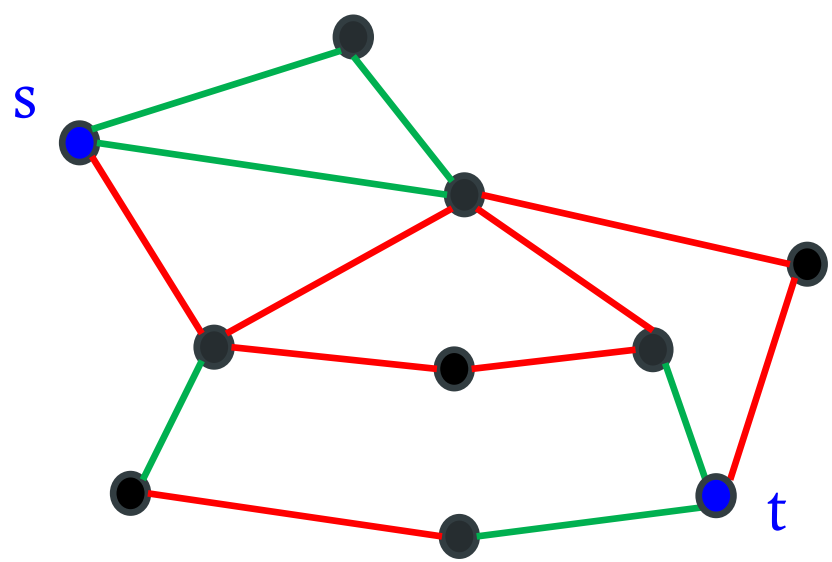

s-t are (2,2) and (1,3) flex-connected

Flexible Connectivity Model

Definition: A pair of vertices (s,t) are (p,q)-flex-connected if s and t are p-edge-connected after removing any set of q unsafe edges

Claim: (s,t) are (p,q)-flex-connected iff for each cut A that separates s from t, $\delta(A)$ has (p+q) edges or has at least p safe edges

(2,3)-flex-connectivity

Flex-SNDP

Input:

- undirected graph $G=(V,E)$, $E = S \uplus U$
- for each pair of nodes (s,t) a *flex-connectivity requirement* $(p(st),\ q(st))$

Goal: *min-cost* subgraph H of G such that each (s,t) is $(p(st),\ q(st))$ flex-connected in H

Flex-SNDP

Goal: *min-cost* subgraph H of G such that each (s,t) is (p(st), q(st)) flex-connected in H

- *All edges are safe*: then same as asking p(st) connectivity for st
- *All edges are unsafe:* then same as asking p(st) + q(st) connectivity for st

Both reduce to SNDP

Why?

Practical:

- perhaps useful model?
- related to other non-uniform fault models

Theoretical:

- related to element connectivity (reliable and unreliable nodes) but quite different too
- related to capacitated network design
- hardness results show problem is harder than SNDP but seems tractable when p, q are fixed which is closer to practice
- standard network design methods don't work easily. testbed for new ideas

Flex-SNDP Special Cases

- Requirement only for *one pair* (s,t) **Flex-ST**
- Spanning requirement (all pairs) **Flex-Spanning**
- Requirement is (0,0) or (p,q) **(p,q)-Flex-SNDP**

Summary of Past Work

- [Adjiashivili'13] defined model and considered (1,k)-Flex-ST and (k,1)-Flex-ST
  - Follow up work by [Adjiashvili,Hommelsheim,Muhlenthaler, Schaudt'20] including hardness results
- [Adjiashvili,Hommelsheim,Muhlenthaler'20] 2.523 approx. for (1,1)-Flex-Spanning
- [Boyd-Cheriyan-Haddadan-Ibrahimpur'21] several results/ideas
  - 2-approx. for (1,1)-Flex-Spanning
  - 4-approx for (k,1)-Flex-Spanning
  - (k+1)-approx. for (1,k)-Flex-Spanning
  - O(q log n) for (p,q)-Flex-Spanning

Summary

- Even (1,k)-Flex-ST and (k,1)-Flex-ST are hard to approx. when k is large
- (k,1)-Flex-SNDP and (1,k)-Flex-SNDP admit O(k) approximation via Capacitated-SNDP
- (p,q)-Flex-Spanning admits an O(q log n) approximation via ideas from Capacitated-SNDP
- (p,q)-Flex-ST does not have a non-trivial approx. when $p, q \geq 2$

Conjecture

Conjecture: There is a poly-time $f(p,q)$ approximation for (p,q)-Flex-SNDP when p, q *are fixed constants*

First non-trivial cases:

(2,2)-Flex-ST and (2,2)-Flex-Spanning

New Results

Via uncrossable+augmentation approach

Single Pair

- 5 approx for (2,2)-Flex-ST
- $f(p,q)$ when $(p+q) > pq/2$, includes $(p,2)$ and $(2,q)$

Spanning

- $2q+2$ approx for $(2, q)$-Flex-Spanning
- $O(p)$ approx for $(p, 2)$ and $(p,3)$-Flex-Spanning

New Results

Via framework of [Chen-LLZ'22] for SNDP based on Raecke trees and group Steiner tree rounding

$O(q(p+q)^3 \log^7 n)$ approx. for (p,q)-Flex-SNDP

As special case of more general result for Bulk Robust model (to be described later)

Related Work

[Bansal-Cheriyan-Grout-Ibrahimpur'23]

- 20 approx. for (p,2)-Flex-Spanning for all p
- Notion of pliable functions that generalize uncrossable functions and a primal-dual algorithm for a special case

Augmentation Approach

Natural and used by [Boyd etal'22] in special cases

- Start with feasible solution F to (p,0) flex-connectivity which is basically an SNDP problem. Can ignore safe/unsafe distinction

- For j = 1 to q do
  - Augment F to satisfy (p,j)-flex-connectivity assuming F satisfied (p,j-1)-flex-connectivity

Augmentation Approach

Augment F to satisfy (p,j)-flex-connectivity assuming F satisfied (p,j-1)-flex-connectivity

Consider a vertex pair (s,t) with requirement (p,j)

Defn: Call a cut $A \subset V$ *deficient* if

- A separates some pair (s,t) with requirement (p,j) and
- $|\delta_F(A)| < p + j$ and
- $|\delta_F(A) \cap S| < p$

Augmentation Approach

Defn: Call a cut $A \subset V$ *deficient* if

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Observations:

- Sufficient to cover all deficient cuts
- *Can ignore safe/unsafe in covering problem*

Augmentation Approach

Defn: Call a cut $A \subset V$ *deficient* if

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Observation:

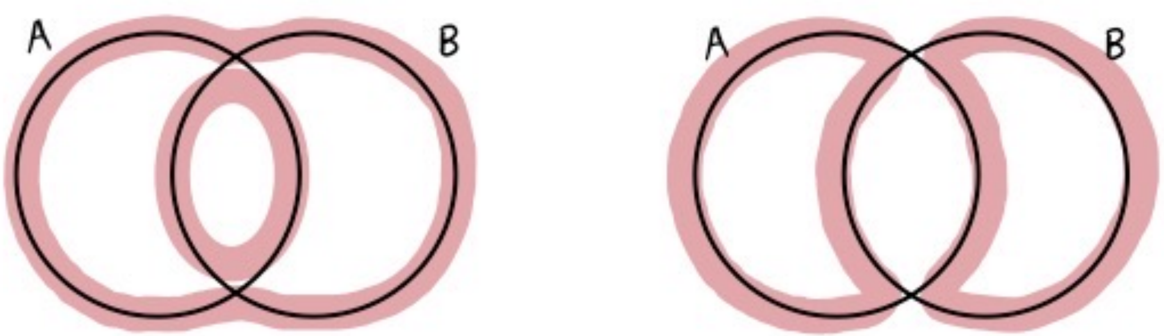
- Sufficient to cover all deficient cuts
- Can ignore safe/unsafe in covering problem

Difficulty: Deficient cuts do not form an *uncrossable family* even in very special cases

Uncrossable Cuts/Function

Defn: A family of cuts C is *uncrossable* if for $A, B \in C$ one of the following is true:

- $A \cap B, A \cup B \in C$
- $A - B, B - A \in C$



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[Williamson-Goemans-Mihail-Vazirani'95]

Theorem: Primal-dual 2-approximation for min-cost covering of uncrossable family of cuts by edges of a graph

(Also via iterated rounding [Jain'00])

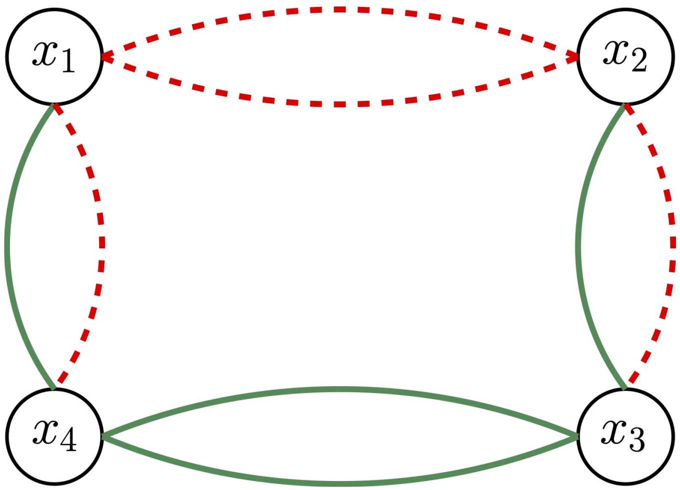
(2,q)-Flex-Spanning

Goal: want *all pairs* to be (2,q)-flex connected

- Start with (2,0)-flex-connectivity (use 2-approx)
- Augmentation from (2,k-1) to (2,k) yields an uncrossable family. Can use 2-approx.
- q stages of augmentation to obtain feasible solution to (2,q)
- Overall 2q+2 approximation

(p,q)-Flex-Spanning

- Want (p,q) for all pairs and p > 2
- Augmentation from (p,k-1) to (p,k) no longer yields uncrossable family
- (3,1) to (3,2)



(p,q)-Flex-Spanning

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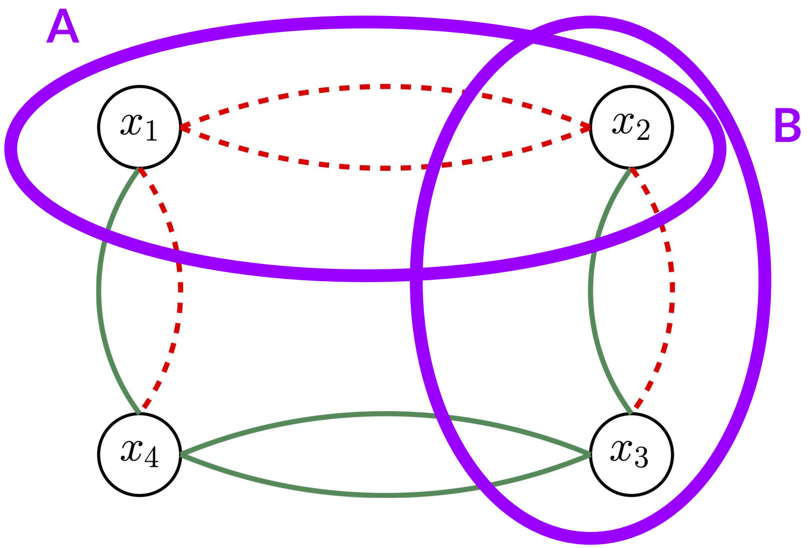

$A \cup B, B - A$ have 3 safe edges and hence *not* deficient

Refining Augmentation Approach

- Partition family of deficient cuts into a *small number* of uncrossable families
- Cover each uncrossable family via the known 2-approximation

(p,2)-Flex-Spanning

2p+4 approx. as follows

- Start with feasible solution F for (p,1)-flex-connectivity (4 approx. known)
- Augment to (p,2). How?
- $\mathcal{C}$ is set of deficient cuts

  { S : $\delta(S)$ has $<$ p safe edges *and* $<$ p+2 total edges }
- $\mathcal{C}$ is not uncrossable

(p,2)-Flex-Spanning

2p+4 approx. as follows

- Start with feasible solution F for (p,1)-flex-connectivity (4 approx. known)
- Augment to (p,2) in *stages*
  - Partition deficient cuts C *into* $C_0, C_1, C_2, \dots, C_{p-1}$
  - C_j is the set of cuts that have exactly j safe edges
  - C_j *is uncrossable* if $C_0, C_1, \dots, C_{j-1}$ are *empty*
  - Cover $C_0, C_1, C_2, \dots, C_{p-1}$ in *sequence*

(p,3)-Flex-Spanning

4p+4 approx. as follows

- Start with feasible solution F for (p,2)-flex-connectivity via 2p+4 approx that we saw
- Augment to (p,3) in *stages*
  - Partition deficient cuts C *into* $C_0, C_1, C_2, \dots, C_{p-1}$
  - C_j is the set of cuts that have exactly j safe edges
  - C_j is uncrossable if $C_0, C_1, \dots, C_{j-1}$ are *empty*
  - Cover $C_0, C_1, C_2, \dots, C_{p-1}$ in *sequence*

(p,4)-Flex-Spanning

Same approach works for augmenting from (p,3) to (p,4) when p is even but *fails* when p is odd

Open: constant factor for (3,4)-Flex-Spanning

(2,2)-Flex-ST

5 approx. for (2,2)-Flex-ST (single pair)

- Start with feasible solution F for (2,1)-flex-connectivity
- **Main Lemma for Augmentation to** (2,2): Deficient cuts can be partitioned into 3 uncrossable families (in fact ring families).

(p,q)-Flex-ST

5 approx. for (2,2)-Flex-ST (single pair)

- Start with feasible solution F for (2,1)-flex-connectivity
- **Main Lemma for Augmentation to (2,2)**: Deficient cuts can be partitioned into 3 uncrossable families (in fact ring families).

(p,q)-Flex-ST

Combine both ideas to obtain

f(p,q) approx. for (p,q)-Flex-ST when $(p+q) > pq/2$

Open: f(p,q) approximation for (p,q)-Flex-ST for all fixed p,q

Bulk Robust Model

[Adjiashvili-Sitters-Zenklusen'2015]

- *Explicitly list* r scenarios
- Each scenario i specifies subset F_i of edges that can fail and a set of vertex pairs K_i
- **Goal:** find cheapest subgraph H of G such that

  For each $i \in [r]$, every pair $(s_j, t_j) \in K_i$ is connected in the graph $H - F_i$

Flexible Connectivity and Bulk Robust

How can we reduce (p,q)-Flex-SNDP to Bulk Robust?

Flexible Connectivity and Bulk Robust

How can we reduce (p,q)-Flex-SNDP to Bulk Robust?

For each subset of at most q unsafe edges and at most (p-1) safe edges a scenario

Want given pairs to be connected in each scenario

Bulk Robust Model

Advantage

- Very expressive and general model. Captures SNDP, Flex-SNDP etc for small values of p,q
- Change in perspective and test bed for ideas
- Positive results could suggest that more general problems tractable

Disadvantage

- Need to explicitly list scenarios which is infeasible in some cases (for instance SNDP with large connectivity)
- Lack of structure, hardness results, running time etc

Bulk Robust Model

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- Each scenario i specifies subset F_i of edges that can fail and set of pairs K_i
- **Goal:** find cheapest subgraph H of G such that

  For each $i \in [r]$, every pair $(s_j, t_j) \in K_i$ is connected in the graph $H - F_i$

Width: $k = \max_i |F_i|$

Corresponds to max connectivity requirement

Known Results

[Adjiashvili-Sitters-Zenklusen'2015]

Spanning: want H with $H - F_i$ is connected for each scenario i

- O(log n + log r) approximation
- Hardness of $\Omega(\log\ r)$ when width is unbounded

Single-Pair: want H with s and t connected in $H - F_i$ for each i

- O(log r) approximation for fixed width
- O(1) approximation when width = 2
- $\Omega(\log r)$ hardness when width is unbounded

New Result

Theorem: Randomized $O(k^4 \log^7 n)$ approximation for Bulk Robust Network Design

Corollary: Randomized $O(q(p+q)^3 \log^7 n)$ approximation for (p,q)-Flex-SNDP

Proof also establishes integrality gap upper bound for natural LP relaxation

LP Relaxation

$$\min \sum_e c_e x_e$$

$$\sum_{e \in \delta(A) - F_i} x_e \geq 1 \qquad i \in [r], A\ separates\ (s_j, t_j) \in K_i$$

$$x_e \geq 0 \qquad e \in E$$

Rounding

- A powerful framework of [Chen-Laekhanukit-Liao-Zhang FOCS'22] designed for higher connectivity version of group Steiner tree/forest (initial ideas from [Grandoni-Laekhanukit-Li'19])
- Uses Raecke trees for oblivious routing plus group Steiner tree/forest oblivious rounding
- Can be adapted to work for Bulk-Robust

Rounding Algorithm

1. Solve LP relaxation to find fractional solution x
2. View x as capacities on graph G
3. Initialize F to be empty set
4. Repeat t times for some parameter t
    1. Sample a random tree (T,y) from Raecke tree distribution for (G,x)
    2. Repeat t' times for some parameter t'
        1. Do *oblivious* random rounding on T for group Steiner connectivity using y as fractional solution
        2. Add to F all edges from preceding step

Improvements

Via another technique the approximation ratio can be improved to $\tilde{O}(\log^2 n)$ for fixed width

Open Problems

Conjecture: There is a poly-time f(p,q) approximation for (p,q)-Flex-SNDP when p, q *fixed constants*

Not known even for single pair or spanning. No lower bounds on integrality gap precluding above

Question: Is there a poly-time f(k) approximation for Bulk Robust Network Design where k is width?

Hardness and **integrality gaps**

Related models and **connections**

Thank You!