Approximate Representation of Symmetric Submodular Functions via Hypergraph Cut Functions

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Abstract

Submodular functions are fundamental to combinatorial optimization. Many interesting problems can be formulated as special cases of problems involving submodular functions. In this work, we consider the problem of approximating symmetric submodular functions everywhere using hypergraph cut functions. Devanur, Dughmi, Schwartz, Sharma, and Singh [5] showed that symmetric submodular functions over n-element ground sets cannot be approximated within $(n/8)$-factor using a graph cut function and raised the question of approximating them using hypergraph cut functions. Our main result is that there exist symmetric submodular functions over n-element ground sets that cannot be approximated within a $o(n^{1/3}/\log^2 n)$-factor using a hypergraph cut function. On the positive side, we show that symmetrized concave linear functions and symmetrized rank functions of uniform matroids and partition matroids can be constant-approximated using hypergraph cut functions.

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1 Introduction

A set function $f : 2^V \to \mathbb{R}_{\geq 0}$ defined over a ground set V is *submodular* if $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ for all subsets $A, B \subseteq V$ and is *symmetric* if $f(A) = f(V - A)$ for all subsets $A \subseteq V$. Submodular functions have the diminishing marginal returns property which arise frequently in economic and game theoretic contexts. Well-known examples of submodular functions include matroid rank functions and graph/hypergraph cut functions. Owing to these connections, submodular functions play a fundamental role in combinatorial optimization.

Throughout this work, we will be interested in non-negative set functions $f : 2^V \to \mathbb{R}_{\geq 0}$ with $f(\emptyset) = 0$. We use n to denote the size of the ground set V. For a parameter $\alpha \geq 1$, a set function $g : 2^V \to \mathbb{R}_{\geq 0}$ is said to α-approximate a set function $f : 2^V \to \mathbb{R}_{\geq 0}$ if

$$g(A) \leq f(A) \leq \alpha g(A) \;\forall\; A \subseteq V.$$

Given the prevalence of submodular functions in combinatorial optimization, a natural question that has been studied is whether an arbitrary submodular set function can be well-approximated by a concisely representable function. We distinguish between structural and algorithmic variants of this question: the structural question asks whether submodular functions can be well-approximated via concisely representable functions while the algorithmic question asks whether such a concise representation can be constructed using polynomial number of function evaluation queries (note that the algorithmic question is concerned with the number of function evaluation queries as opposed to run-time). Concise representations with small-approximation factor are useful in learning, testing, streaming, and sketching algorithms. Consequently, concise representations with small-approximation factor for submodular functions (and their generalizations and subfamilies of submodular functions) have been studied from all these perspectives with most results focusing on monotone submodular functions [8, 2, 12, 1, 5, 11, 6, 3].

In this work, we focus on approximating *symmetric* submodular functions. Balcan, Harvey, and Iwata [2] showed that for every symmetric submodular function $f : 2^V \to \mathbb{R}_{\geq 0}$, there exists a function $g : 2^V \to \mathbb{R}_{\geq 0}$ defined by $g(S) := \sqrt{\chi(S)^T M \chi(S)}$, where $\chi(S) \in \{0,1\}^V$ is the indicator vector of $S \subseteq V$ and M is a symmetric positive definite matrix such that g $\sqrt{n}$-approximates f. We note that such a function g has a concise representation—namely, the matrix M. Is it possible to improve on the approximation factor for symmetric submodular functions using other concisely representable functions?

The concisely representable family of functions that we study in this work is the family of hypergraph cut functions. A hypergraph $H = (V, E)$ consists of a vertex set V and hyperedges E where each hyperedge $e \in E$ is a subset of vertices. If every hyperedge has size 2, then the hypergraph is simply a graph. For a subset A of vertices, we use $\delta(A)$ to denote the set of hyperedges e such that e has non-empty intersection with both A and $V \setminus A$. The cut function $d : 2^V \to \mathbb{R}_+$ of a hypergraph $H = (V, E)$ with hyperedge weights $w : E \to \mathbb{R}_+$ is given by

$$d(A) := \sum_{e \in E:\ e \in \delta(A)} w_e \ \forall\ A \subseteq V.$$

A function $g : 2^V \to \mathbb{R}_+$ is a hypergraph cut function if there exists a weighted hypergraph with vertex set V whose cut function is g. We will say that a function $f : 2^V \to \mathbb{R}_+$ is *α-hypergraph-approximable (α-graph approximable)* if there exists a hypergraph (graph) cut function g such that g α-approximates f. We note that although a hypergraph could have exponential number of hyperedges, every n-vertex hypergraph admits a $(1+\epsilon)$-approximate cut-sparsifier with $O(\frac{n \log n}{\epsilon^2})$ hyperedges (see Theorem 6 for a formal definition of cut-sparsifier), and hence, hypergraph cut functions have a concise representation (with a constant loss in approximation factor).

The structural approximation question of whether every symmetric submodular function is constant-hypergraph-approximable was raised by Devanur, Dughmi, Shwartz, Sharma, and Singh [5]. They showed that every symmetric submodular function on a ground set of size n is $O(n)$-graph-approximable and that this factor is tight for graph-approximability: in fact, the cut function of the n-vertex hypergraph containing a single hyperedge that contains all vertices cannot be $(n/4 - \epsilon)$-approximated by a graph cut function for all constant $\epsilon > 0$. This example naturally raises the following intriguing conjecture:

▶ **Conjecture 1.** *Every symmetric submodular function is constant-hypergraph-approximable.*

The conjecture is further fueled by the fact that there are no natural examples of symmetric submodular functions besides hypergraph cut functions (although arbitrary submodular functions can be symmetrized while preserving submodularity).

We emphasize that the algorithmic variant of Conjecture 1 is false. In particular, there does not exist an algorithm that makes a polynomial number of function evaluation queries to a symmetric submodular function f and constructs a hypergraph cut function g such that g $O(\sqrt{n/\ln n})$-approximates f. We outline a proof of this observation now. Suppose that there exists an algorithm that uses polynomial number of function evaluation queries to a given symmetric submodular function f to construct a weighted hypergraph whose cut function α-approximates f; then we can obtain an α-approximation to the *symmetric submodular sparsest cut problem* by constructing such a hypergraph and solving the sparsest cut on that hypergraph exactly (using exponential run-time). However, Svitkina and Fleischer [12] have shown that the best possible approximation for the symmetric submodular sparsest cut problem using polynomial number of function evaluation queries is $\Omega(\sqrt{n/\ln n})$ (even if exponential run-time is allowed). Hence, the algorithmic version of hypergraph-aproximability has a strong lower bound of $\Omega(\sqrt{n/\ln n})$. This leaves the structural question open while perhaps, hinting that it may also have a strong lower bound.

1.1 Our Results

The symmetrization of a set function $f : 2^V \to \mathbb{R}$ is the function $f_{\text{sym}} : 2^V \to \mathbb{R}$ obtained as

$$f_{\text{sym}}(A) := f(A) + f(V \setminus A) - f(V) - f(\emptyset).$$

We note that if $f : 2^V \to \mathbb{R}$ is submodular, then its symmetrization $f_{\text{sym}} : 2^V \to \mathbb{R}$ is symmetric submodular. A matroid rank function is a non-negative integer valued submodular set function $r : 2^V \to \mathbb{Z}$ satisfying $r(A) \le r(A \cup \{e\}) \le r(A) + 1$ for every subset $A \subseteq V$ and element $e \in V$. As a step towards understanding Conjecture 1, we observe that it suffices to focus on symmetrized matroid rank functions (see Section 1.3 for a proof).

▶ **Proposition 2.** *If the symmetrization of every matroid rank function is α-hypergraph-approximable, then every rational-valued symmetric submodular function is α-hypergraph-approximable.*

Next, we refute Conjecture 1 by showing the following result.

▶ **Theorem 3.** *For every sufficiently large positive integer n, there exists a matroid rank function $r : 2^{[n]} \to \mathbb{Z}_{\ge 0}$ such that r_{sym} is not α-hypergraph-approximable for*

$$\alpha = o\left(\frac{n^{\frac{1}{3}}}{\log^2 n}\right).$$

Our proof of Theorem 3 is an existential argument and it does not construct an explicit matroid rank function that achieves the lower bound.

Next, we prove positive approximation results for certain subfamilies of symmetric submodular functions. The subfamilies that we consider are inspired by Proposition 2 and by previous work on approximating symmetric submodular functions and matroid rank functions.

We call a set function $f : 2^V \to \mathbb{R}_{\ge 0}$ as a *concave linear function* if there exist weights $w : V \to \mathbb{R}_{\ge 0}$ and an increasing concave function $h : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that $f(S) = h(\sum_{v \in S} w_v)$ for every $S \subseteq V$. We note that concave linear functions are submodular. Goemans, Harvey, Iwata, and Mirrokni [8] showed that every matroid rank function over a n-element ground set can be $\sqrt{n}$-approximated by the square-root of a linear function, i.e., by a concave linear function. Balcan, Harvey and Iwata [2] showed that every symmetric submodular function $f : 2^V \to \mathbb{R}_{\geq 0}$ is $\sqrt{n}$-approximated by a function $g : 2^V \to \mathbb{R}_{\geq 0}$ of the form $g(S) := \sqrt{\chi(S)^T M \chi(S)}$ for all $S \subseteq V$, where $\chi(S) \in \{0,1\}^V$ is the indicator vector of S and M is a symmetric positive definite matrix. In particular, if M is a diagonal matrix, then the function g is the square root of a linear function, i.e., a concave linear function. Given the significant role of concave linear functions, we consider the hypergraph-approximability of such functions.

▶ **Theorem 4.** *Symmetrized concave linear functions are* 128*-hypergraph-approximable.*

As a special case of Theorem 4, we obtain that the symmetrized rank function of uniform matroids is constant-hypergraph-approximable. Thus, symmetrized rank functions of uniform matroids act as a starting point for identifying subfamilies of symmetrized matroid rank functions that are constant-hypergraph-approximable. We consider a generalization of the uniform matroid, namely the partition matroid and show that it is also constant-hypergraph-approximable. We refer the reader to Section 1.2 for formal definitions of uniform and partition matroids.

▶ **Theorem 5.** *Symmetrized rank functions of uniform matroids and partition matroids are* 64*-hypergraph-approximable.*

Theorem 5 gives a concrete class of functions for which there is a large gap between the approximation capabilities of graph cut functions and hypergraph cut functions. Consider the uniform matroid where the independent sets are those of size at most 1. The symmetrized rank function of this matroid is the same as the cut function of a hypergraph with a single hyperedge spanning all vertices. As mentioned above, this function cannot be $(n/4 - \epsilon)$-approximated by a graph cut function for all constant $\epsilon > 0$ [5]. Thus, symmetrized rank functions of uniform and partition matroids cannot be better than $n/4$ approximated by graph cut functions, but can be constant factor approximated by hypergraph cut functions.

While our lower bound result in Theorem 3 rules out α-hypergraph-approximability for symmetric submodular functions for $\alpha = o(n^{1/3}/\log^2 n)$, our positive results suggest broad families of symmetric submodular functions which are constant-hypergraph-approximable. It would be interesting to characterize the family of symmetric submodular functions that are constant-hypergraph-approximable. We also do not know if our lower bound result in Theorem 3 is tight. We only know that every symmetric submodular function is $(n-1)$-graph-approximable. It would be interesting to show that every symmetric submodular function is $\tilde{O}(n^{1/3})$-hypergraph-approximable—we believe that Proposition 2 and Theorem 4 should help towards achieving this approximation factor.

1.2 Preliminaries

A set function $f : 2^V \to \mathbb{R}_{\geq 0}$ is submodular if $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ for all subsets $A, B \subseteq V$, symmetric if $f(A) = f(V - A)$ for all subsets $A \subseteq V$, and monotone if $f(B) \geq f(A)$ for all subsets $A \subseteq B \subseteq V$.

A matroid $\mathcal{M} = (V, \mathcal{I})$ is specified by a ground set V and a collection $\mathcal{I} \subseteq 2^V$, known as independent sets, satisfying the three independent set axioms: (1) $\emptyset \in \mathcal{I}$, (2) if $B \in \mathcal{I}$, then $A \in \mathcal{I}$ for every $A \subseteq B$, and (3) if $A, B \in \mathcal{I}$ with $|B| > |A|$, then there exists an element $v \in B \setminus A$ such that $A \cup \{v\} \in \mathcal{I}$. The rank function $r : 2^V \to \mathbb{Z}_{\geq 0}$ of a matroid $\mathcal{M} = (V, \mathcal{I})$ is defined as

$$r(A) := \max\{|S| : S \subseteq A,\ S \in \mathcal{I}\}\ \forall\ A \subseteq V.$$

The definition of matroid rank functions that we presented in Section 1.1 is equivalent to this definition [10]. It is well-known that the rank function of a matroid is monotone submodular.

We consider two matroids over the ground set V. A uniform matroid is a matroid in which the independent sets are exactly the sets containing at most k elements of the ground set V, for some fixed integer k—we call it as the uniform matroid over ground set V with budget k. Partition matroids generalize uniform matroids: the independent sets of the partition matroid associated with a partition $P_1, \ldots, P_t$ of the ground set V with budgets $b_1, \ldots, b_t \in \mathbb{Z}_{\geq 0}$ are those subsets $A \subseteq V$ for which $|A \cap P_i| \leq b_i$ for every $i \in [t]$.

The proof of our lower bound will use the following theorem showing the existence of cut-sparsifiers.

▶ **Theorem 6.** *[4] For every positive constant ϵ and for every weighted n-vertex hypergraph H, there exists another weighted hypergraph H' (called a cut-sparsifier) on the same vertex set with $\tilde{O}(n/\epsilon^2)$ hyperedges such that the cut function of H' $(1+\epsilon)$-approximates the cut function of H.*

1.3 Proof of Proposition 2

In this section, we prove Proposition 2. We need the notion of contraction of set functions and hypergraphs. For a set function $f : 2^V \to \mathbb{R}_{\geq 0}$ and a subset A, the function $g : 2^{V-A+a} \to \mathbb{R}_{\geq 0}$ obtained by contracting f with respect to A is defined as

$$g(S) := \begin{cases} f(S) \text{ if } a \notin S, \\ f(S-a+A) \text{ if } a \in S. \end{cases}$$

If $f : 2^V \to \mathbb{R}_{\geq 0}$ is a symmetric submodular function and $A \subseteq V$, then the function obtained by contracting f with respect to A is also symmetric submodular. Let $H = (V, E)$ be a hypergraph with hyperedge weights $w : E \to \mathbb{R}_{\geq 0}$ and $A \subseteq V$. Then, the hypergraph obtained by contracting H with respect to A is defined as $H' = (V - A + a, E')$ where a is a new vertex not present in V and

$$E' := \{e - A + a : e \in E, e \cap A \neq \emptyset\} \cup \{e : e \in E, e \cap A = \emptyset\}.$$

We note that E' could have self-loops and that there is a surjection $\phi : E \to E'$ mapping each hyperedge to the hyperedge it is contracted into (which could be the same as the original hyperedge). We use this surjection to define the weight $w' : E' \to \mathbb{R}_{\geq 0}$ of hyperedges in E' as $w'(e') = \sum_{e \in E:\, \phi(e) = e'} w(e)$. We note that if f is the cut function of a weighted hypergraph $H = (V, E)$ with hyperedge weights $w : E \to \mathbb{R}_{\geq 0}$ and $A \subseteq V$, then the contraction of f with respect to A corresponds to the cut function of the weighted hypergraph (H', w') obtained by contracting H with respect to A. This leads to the following observation:

▶ **Observation 7.** *The contraction of a α-hypergraph-approximable function $f : 2^V \to \mathbb{R}_{\geq 0}$ with respect to a subset $A \subseteq V$ is also α-hypergraph-approximable.*

▶ **Proposition 2.** *If the symmetrization of every matroid rank function is α-hypergraph-approximable, then every rational-valued symmetric submodular function is α-hypergraph-approximable.*

Proof. It suffices to consider integer-valued symmetric submodular functions (multiply all function values by the product of the denominators of their rational expressions). Hence, we will focus on approximating integer-valued symmetric submodular functions.

Let $f : 2^V \to \mathbb{Z}_{\geq 0}$ be an integer-valued symmetric submodular function. It is known that there exists a vector $w \in \mathbb{R}^V$ such that the function $g : 2^V \to \mathbb{R}_{\geq 0}$ defined by

$$g(S) := f(S) + \sum_{u \in S} w_u \;\forall\; S \subseteq V$$

is integer-valued, monotone, and submodular [7, Section 3.3] (e.g., for our purposes, we can simply choose $w_u := \max\{f(S) \colon S \subseteq V\}$ for every $u \in V$). Since $f(V) = 0$, we have that $g(V) = \sum_{u \in V} w_u$. Consequently, $(1/2)g_{\text{sym}}(S) = (1/2)(g(S) + g(V - S) - g(V)) = (1/2)(f(S) + f(V - S)) = f(S)$ for every $S \subseteq V$ since f is symmetric. Thus, $f(S) = (1/2)g_{\text{sym}}(S)$ for every $S \subseteq V$.

Next, consider the integer-valued monotone submodular function $g : 2^V \to \mathbb{Z}_{\geq 0}$ obtained as above. Helgason [9] showed that there exists a matroid on a ground set U with rank function $r : 2^U \to \mathbb{Z}_{\geq 0}$ and a partition $(U_v : v \in V)$ of U such that $g(S) = r(\cup_{v \in S} U_v)$ for every $S \subseteq V$. Equivalently, the function g is obtained from the rank function r by repeatedly contracting with respect to U_v for each $v \in V$ (the order of processing $v \in V$ is irrelevant). Moreover, $f(S) = (1/2)g_{\text{sym}}(S) = (1/2)r_{\text{sym}}(\cup_{v \in S} U_v)$ for every $S \subseteq V$. Hence, the function f is half times the contraction of a symmetrized matroid rank function. Thus, if every symmetrized matroid rank function is α-hypergraph-approximable, then by Observation 7, the function f is also α-hypergraph-approximable. ◀

2 Lower Bound

In this section, we prove Theorem 3. In our first lemma, we show that it suffices to consider only hypergraphs with $\tilde{O}(n)$ hyperedges if we are willing to tolerate a constant loss in the approximation factor.

▶ **Lemma 8** (Few hyperedges suffice). *Let $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be a symmetric submodular function and $\beta \geq 1$ be a positive real number. Suppose that there exists a weighted hypergraph H whose cut function β-approximates f. Then, there exists a weighted hypergraph H' with $\tilde{O}(n)$ hyperedges whose cut function 2β-approximates f.*

Proof. Applying Theorem 6 to H with $\epsilon = 1$ gives us that there exists a weighted hypergraph H' with $\tilde{O}(n)$ hyperedges whose cut function 2-approximates the cut function of H. Since the cut function of H β-approximates f, this means that the cut function of H' 2β-approximates f. ◀

Next we show that it suffices to restrict our attention to hypergraphs with rational hyperedge weights while again losing only a constant in the approximation factor (since we will be considering only hypergraphs with $O(n)$ hyperedges).

▶ **Lemma 9** (Bounded rational weights suffice). *Let $r : 2^V \to \mathbb{R}_+$ be a matroid rank function on ground set $V = [n]$. Suppose that there exists a hypergraph $H = (V, E)$ with hyperedge weights $w \colon E \to \mathbb{R}_+$ with $|E| = \tilde{O}(n)$ whose cut function $d : 2^{[n]} \to \mathbb{R}_{\geq 0}$ β-approximates r_{sym} for some $\beta = o(n)$. Then, there exist hyperedge weights $w' \colon E \to \mathbb{Q}_+$ which assign to each hyperedge of H a positive rational weight p/q where $p, q \leq n^3$ such that d' 2β-approximates r_{sym}, where $d' : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is the cut function induced by the weight function w'.*

Proof. Since r is the rank function of a matroid on ground set $V = [n]$, we have that $r(S) \leq n$ for all $S \subseteq [n]$, and therefore $r_{\text{sym}}(S) \leq n$ for all $S \subseteq [n]$. Consequently, if $w(e) > n$ for some $e \in E$ then we have $d(S) > n \geq r_{\text{sym}}(S)$ for some $S \subseteq [n]$, a contradiction. Thus, we conclude that $w(e) \leq n$ for every $e \in E$.

We define the new weight function $w' : E \to \mathbb{R}_{\geq 0}$ by

$$w'(e) := \frac{\lfloor n^2 w(e) \rfloor}{n^2} \; \forall \, e \in E.$$

For every $e \in E$, the weight $w'(e)$ is a rational number p/q with $q = n^2$ and $p \leq n^2 w(e) \leq n^3$. Next, we show that the cut function $d' : 2^{[n]} \to \mathbb{R}_{\geq 0}$ induced by this weight function w' satisfies the required bounds for every subset $S \subseteq [n]$.

For every $e \in E$, we have that $w'(e) \leq w(e)$. Thus, for every $S \subseteq [n]$, we have that $d'(S) \leq d(S) \leq r_{\text{sym}}(S)$. Moreover, for every $e \in E$, we have that $w'(e) \geq w(e) - 1/n^2$. Therefore, for every $S \subseteq [n]$, we have that

$$d'(S) \geq d(S) - |E|/n^2. \tag{1}$$

Let $S \subseteq [n]$. If $r_{\text{sym}}(S) = 0$, then $d'(S) \leq d(S) = 0$, and so $r_{sym}(S) \leq 2\beta d'(S)$. Suppose $r_{\text{sym}}(S) > 0$. Since $r_{\text{sym}}(S)$ is an integer, this means that $r_{\text{sym}}(S) \geq 1$, and therefore $1/\beta \leq r_{\text{sym}}(S)/\beta \leq d(S)$. Since $|E| = \tilde{O}(n)$, we have that $|E|/n^2 = \tilde{O}(\frac{1}{n})$, and since $\beta = o(n)$, we conclude that $|E|/n^2 < 1/2\beta \leq d(S)/2$. Hence, Inequality (1) gives us that $d'(S) \geq d(S)/2$, and therefore, $r_{\text{sym}}(S) \leq \beta d(S) \leq 2\beta d'(S)$.

◀

We will show the existence of our desired matroid rank function using the following theorem of Balcan and Harvey [3].

▶ **Theorem 10.** *[3] For every positive integer n and $k \geq 8$ with $k = 2^{o(n^{1/3})}$, there exists a family of sets $\mathcal{A} \subseteq 2^{[n]}$ and a family of matroids $\mathcal{M} = \{M_{\mathcal{B}} : \mathcal{B} \subseteq \mathcal{A}\}$ on the ground set $[n]$ with the following properties:*

1. *$|\mathcal{A}| = k$ and $|A| = \lfloor n^{1/3} \rfloor$ for every $A \in \mathcal{A}$.*
2. *For every $\mathcal{B} \subseteq \mathcal{A}$ and every $A \in \mathcal{A}$, we have*

$$rank_{M_{\mathcal{B}}}(A) = \begin{cases} 8\lfloor \log k \rfloor & (\textit{if } A \in \mathcal{B}) \\ |A| = \lfloor n^{1/3} \rfloor & (\textit{if } A \in \mathcal{A} \setminus \mathcal{B}) \end{cases}.$$

3. *For every $A_1, A_2 \in \mathcal{A}$ with $A_1 \neq A_2$, we have $|A_1 \cap A_2| \leq 4 \log k$.*
4. *For every $\mathcal{B} \subsetneq \mathcal{A}$, we have $rank_{M_{\mathcal{B}}}([n]) = \lfloor n^{1/3} \rfloor$.*

We note that the version of the theorem given in [3] does not include the third and fourth properties. However, the proof for the variant of Theorem 10 with the first two properties given in [3] shows that the third and fourth properties also hold. We are now ready to prove Theorem 3. The following is a restatement of Theorem 3.

▶ **Theorem 11.** *For every sufficiently large positive integer n, there exists a symmetrized matroid rank function $r_{sym} : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ on ground set $[n]$ such that r_{sym} is not α-hypergraph-approximable for $\alpha = o(n^{1/3}/\log^2 n)$.*

Proof. For simplicity, we will assume that $n = 8^x$ for some positive integer x, so that $\log n$ and $n^{1/3}$ are both integers. If the theorem holds for n of this form, it holds for all sufficiently large n, since for any $8^x \leq n < 8^{x+1}$ we can extend a matroid M on ground set $[8^x]$ to a matroid M' on ground set $[n]$ which has the same independent sets.

For $k = n^{\log n}$, let $\mathcal{A}$ be a collection of subsets of $[n]$ and $\mathcal{M} = \{M_{\mathcal{B}} : \mathcal{B} \subseteq \mathcal{A}\}$ be the family of matroids on ground set $[n]$ with the properties guaranteed by Theorem 10. We note that $|\mathcal{M}| = 2^{n^{\log n}}$. For each $\mathcal{B} \subseteq \mathcal{A}$, let $r^{\mathcal{B}}_{\text{sym}}$ be the symmetrized rank function of $M_{\mathcal{B}}$ and let

$\mathcal{F} := \{r^{\mathcal{B}}_{\text{sym}} \colon M_{\mathcal{B}} \in \mathcal{M}\}$ be the family of symmetrized rank functions of matroids in the family $\mathcal{M}$. We note that $\mathcal{F}$ is a family of $2^{n^{\log n}}$ symmetrized matroid rank functions over the ground set $[n]$. We will prove that there exists $r^{\mathcal{B}}_{\text{sym}} \in \mathcal{F}$ which is not α-hypergraph-approximable. Suppose for contradiction that for every function $r^{\mathcal{B}}_{\text{sym}} \in \mathcal{F}$ there exists a hypergraph $H_{\mathcal{B}}$ such that the cut function $d_{\mathcal{B}}$ of $H_{\mathcal{B}}$ satisfies $d_{\mathcal{B}}(S) \leq r^{\mathcal{B}}_{\text{sym}}(S) \leq \alpha(n) d_{\mathcal{B}}(S)$ for all $S \subseteq [n]$.

Let $r^{\mathcal{B}}_{\text{sym}} \in \mathcal{F}$. By Lemma 8, there exists a weighted hypergraph $H'_{\mathcal{B}}$ with $\tilde{O}(n)$ hyperedges such that its cut function $d' : 2^{[n]} \to \mathbb{R}_{\geq 0}$ satisfies

$$d'(S) \leq r^{\mathcal{B}}_{\text{sym}}(S) \leq 2\alpha d'(S) \ \forall\ S \subseteq [n].$$

Applying Lemma 9 to the rank function $\text{rank}_{\mathcal{B}} : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ of the matroid $M_{\mathcal{B}}$ and the hypergraph $H'_{\mathcal{B}}$ gives a hypergraph $H''_{\mathcal{B}}$ with $\tilde{O}(n)$ hyperedges all of whose weights are rational values p/q with $p, q \leq n^3$ such that the cut function $d'' : 2^{[n]} \to \mathbb{R}_{\geq 0}$ of $H''_{\mathcal{B}}$ satisfies

$$d''(S) \leq r^{\mathcal{B}}_{\text{sym}}(S) \leq 4\alpha d''(S) \ \forall\ S \subseteq [n].$$

Let $\mathcal{H}$ be the family of weighted hypergraphs $\{H''_{\mathcal{B}} : r^{\mathcal{B}}_{\text{sym}} \in \mathcal{F}\}$.

We now count the number of weighted hypergraphs in $\mathcal{H}$. Each hypergraph in $\mathcal{H}$ has $\tilde{O}(n)$ hyperedges with each hyperedge having rational weight p/q where $p, q \leq n^3$. The number of potential hyperedges in a n-vertex hypergraph is $2^n - 1$, so for every $m \in \mathbb{Z}_+$ the number of simple n-vertex hypergraphs with m hyperedges is $\binom{2^n-1}{m} = O(2^{nm})$. Consequently, the number of possible simple hypergraphs with $\tilde{O}(n)$ hyperedges is $2^{\tilde{O}(n^2)}$. The number of positive rational numbers p/q with $p, q \in [n^3]$ is at most n^6, so the number of ways to assign a weight of this kind to each hyperedge of a hypergraph with $\tilde{O}(n)$ hyperedges is $n^{\tilde{O}(n)}$. Therefore the number of hypergraphs with $\tilde{O}(n)$ hyperedges each of which has a positive rational weight p/q where $p, q \in [n^3]$ is $2^{\tilde{O}(n^2)} n^{\tilde{O}(n)} = 2^{\tilde{O}(n^2)} = 2^{o(n^{\log n})}$. Hence, $|\mathcal{H}| = 2^{o(n^{\log n})}$.

Let $\mathcal{F}' := \{r^{\mathcal{B}}_{\text{sym}} \in \mathcal{F} \colon |B| \leq |\mathcal{A}| - 2\}$. Since $|\mathcal{F}| = 2^{n^{\log n}}$ and $|\mathcal{A}| = n^{\log n}$, we have that $|\mathcal{F}'| = \Omega(2^{n^{\log n}})$. Since $|\mathcal{F}'| = \Omega(2^{n^{\log n}})$ while $|\mathcal{H}| = 2^{o(n^{\log n})}$, there must exist two distinct functions $r^{\mathcal{B}_1}_{\text{sym}}, r^{\mathcal{B}_2}_{\text{sym}} \in \mathcal{F}'$ such that there is a single weighted hypergraph $H \in \mathcal{H}$ whose cut function $d : 2^{[n]} \to \mathbb{R}_{\geq 0}$ satisfies

$$d(S) \leq r^{\mathcal{B}_1}_{\text{sym}}(S) \leq 8\alpha d(S) \ \forall\ S \subseteq [n] \text{ and} \tag{2}$$

$$d(S) \leq r^{\mathcal{B}_2}_{\text{sym}}(S) \leq 8\alpha d(S) \ \forall\ S \subseteq [n]. \tag{3}$$

Since $\mathcal{B}_1 \neq \mathcal{B}_2$, at least one of $\mathcal{B}_1 \setminus \mathcal{B}_2$ and $\mathcal{B}_2 \setminus \mathcal{B}_1$ must be non-empty. We assume without loss of generality that $\mathcal{B}_1 \setminus \mathcal{B}_2 \neq \emptyset$. Let $S \in \mathcal{B}_1 \setminus \mathcal{B}_2$. By Theorem 10, we have that $\text{rank}_{M_{\mathcal{B}_1}}(S) = 8\log^2 n$ and $\text{rank}_{M_{\mathcal{B}_2}}(S) = n^{1/3}$. Since $r^{\mathcal{B}_1}_{\text{sym}}, r^{\mathcal{B}_2}_{\text{sym}} \in \mathcal{F}'$, we have that $|\mathcal{B}_1|, |\mathcal{B}_2| \leq |\mathcal{A}| - 2$, and thus $|\mathcal{B}_1 \cup \{S\}|, |\mathcal{B}_2 \cup \{S\}| \leq |\mathcal{A}| - 1$. Therefore $\mathcal{A} \setminus (\mathcal{B}_1 \cup \{S\}), \mathcal{A} \setminus (\mathcal{B}_2 \cup \{S\}) \neq \emptyset$, so there exist sets $T_1 \in \mathcal{A} \setminus (\mathcal{B}_1 \cup \{S\}), T_2 \in \mathcal{A} \setminus (\mathcal{B}_2 \cup \{S\})$. By Theorem 10, we have that $\text{rank}_{M_{\mathcal{B}_1}}(T_1), \text{rank}_{M_{\mathcal{B}_2}}(T_2) = n^{1/3}$ and $|S \cap T_1|, |S \cap T_2| \leq 4\log^2 n$. Therefore, $\text{rank}_{M_{\mathcal{B}_1}}(T_1 \setminus S), \text{rank}_{M_{\mathcal{B}_2}}(T_2 \setminus S) \geq n^{1/3} - 4\log^2 n$, and so $\text{rank}_{M_{\mathcal{B}_1}}([n] \setminus S), \text{rank}_{M_{\mathcal{B}_2}}([n] \setminus S) \geq n^{1/3} - 4\log^2 n$. Furthermore, since $T_1, T_2 \subseteq [n]$ we have that $\text{rank}_{M_{\mathcal{B}_1}}([n]), \text{rank}_{M_{\mathcal{B}_2}}([n]) \geq n^{1/3}$, and so by Theorem 10, we have $\text{rank}_{M_{\mathcal{B}_1}}([n]), \text{rank}_{M_{\mathcal{B}_2}}([n]) = n^{1/3}$. Thus, we have that

$$\begin{aligned}
r^{\mathcal{B}_1}_{\text{sym}}(S) &= \text{rank}_{M_{\mathcal{B}_1}}(S) + \text{rank}_{M_{\mathcal{B}_1}}([n] \setminus S) - \text{rank}_{M_{\mathcal{B}_1}}([n]) \\
&\leq 8\log^2 n + n^{1/3} - n^{1/3} = 8\log^2 n \quad \text{and} \\
r^{\mathcal{B}_2}_{\text{sym}}(S) &= \text{rank}_{M_{\mathcal{B}_2}}(S) + \text{rank}_{M_{\mathcal{B}_2}}([n] \setminus S) - \text{rank}_{M_{\mathcal{B}_2}}([n]) \\
&\geq n^{1/3} + (n^{1/3} - 4\log^2 n) - n^{1/3} = n^{1/3} - 4\log^2 n.
\end{aligned}$$

Therefore, by inequalities (2) and (3), we have that $d(S) \le r^{\mathcal{B}_1}_{\text{sym}}(S) \le 8\log^2 n$, and $8\alpha d(S) \ge r^{\mathcal{B}_2}_{\text{sym}}(S) \ge n^{1/3} - 4\log^2 n$. Hence, $\alpha = \Omega(n^{1/3}/\log^2 n)$. This contradicts the assumption that $\alpha = o(n^{1/3}/\log^2 n)$. ◀

3 Upper Bounds

In this section, we show that certain subfamilies of symmetric submodular functions are constant-hypergraph-approximable. In particular, we show how to approximate concave linear functions and symmetrized rank functions of uniform and partition matroids using hypergraph cut functions.

3.1 Concave Functions

In this section, we prove Theorem 4. We recall that a set function $f : 2^V \to \mathbb{R}_{\ge 0}$ is a concave linear function if there exist weights $w : V \to \mathbb{R}_{\ge 0}$ and an increasing concave function $h : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that $f(S) = h(\sum_{v\in S} w_v)$. If all weights are one , then f_{sym} is symmetric submodular and moreover, the precise value of $f(S)$ depends only on the size $|S|$ and does not depend on the precise identify of the elements in S, so we call such functions f as anonymized concave linear functions. In Section 3.1.1, we consider the special case of anonymized concave linear functions and show that these are constant-hypergraph-approximable. We extend these ideas in Section 3.1.2 to show that symmetrized concave linear functions are constant-hypergraph-approximable.

3.1.1 Anonymized Concave Linear Functions

The following lemma is useful for proving the main theorem of this section. Its proof is given in the appendix.

▶ **Lemma 12.** *For every integer $n \ge 2$, $r \in \{2, \dots, n\}$, and $X \subseteq [n]$ with $1 \le |X| \le \frac{n}{2}$, the set of hyperedges $\delta(X)$ that cross X in a complete r-uniform n-vertex hypergraph has the following size bound:*

$$\frac{1}{4}\min\left\{\frac{|X|r}{n}, 1\right\} \le \frac{|\delta(X)|}{\binom{n}{r}} \le 4\min\left\{\frac{|X|r}{n}, 1\right\}.$$

The following is the main theorem of this section.

▶ **Theorem 13.** *Let n be a positive real number and $h\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a function such that h is concave on $[0, n]$ and $h(x) = h(n-x)$ for every $x \in [0, n]$. Then, the symmetric submodular function $f : 2^V \to \mathbb{R}_{\ge 0}$ over the ground set $V = [n]$ defined by $f(S) := h(|S|)\ \forall\ S \subseteq V$ is 64-hypergraph-approximable.*

Proof. To simplify our notation, we define $a_x := h(x) - h(x-1)$ for $x \in \{1, \dots, \lceil n/2\rceil\}$. A hypergraph is uniform if all its hyperedges have the same size and a complete t-uniform hypergraph consists of all hyperedges of size t. We define H as the union of $\lceil n/2\rceil$ different hypergraphs, $G_0, \dots, G_{\lceil n/2\rceil}$, each of which is a uniform hypergraph over the vertex set V and each of whose hyperedges are weighted uniformly. Formally, H is the union of:

1. A complete $\lceil \frac{n}{x}\rceil$-uniform hypergraph G_x, with a total weight of $(a_x - a_{x+1})(x/8)$ equally distributed among its hyperedges for each $x \in \{1, \dots, \lceil n/2\rceil - 1\}$, i.e., $w(e) = (a_x - a_{x+1})(x/8)/\binom{n}{\lceil \frac{n}{x}\rceil}$ for every hyperedge $e \in E(G_x)$ (we note that $a_x - a_{x+1} \ge 0$ since h is concave).

2. A complete 2-uniform hypergraph $G_{\lceil n/2 \rceil}$, with a total weight of $a_{\lceil n/2 \rceil}(n/32)$ equally distributed among its hyperedges.
3. A hypergraph G_0 consisting of a single n-vertex hyperedge of weight $h(0)/64$.

Let d be the cut function of the hypergraph H we have just defined. In order to show that d 64-approximates f, we will consider an arbitrary subset C of size k and bound its cut value in H. Since we know that d and f are both symmetric, we assume without loss of generality that $1 \leq k \leq n/2$.

We now compute the weight of hyperedges crossing C in H. We recall that $|C| = k \leq n/2$. We begin with the easy cases. $\delta(C)$ will certainly cut the single hyperedge of G_0 for a weight of exactly $h(0)/64$. The hyperedges in $G_{\lceil n/2 \rceil}$ have rank 2. Therefore, by Lemma 12, the number of hyperedges crossing C in $G_{\lceil n/2 \rceil}$ is at least a $\frac{k}{2n}$ fraction and at most a $\frac{8k}{n}$ fraction of the hyperedges in $G_{\lceil n/2 \rceil}$, for a total weight between $a_{\lceil n/2 \rceil}k/64$ and $a_{\lceil n/2 \rceil}k/4$.

Next, we compute the weight of hyperedges crossing C in $G_1, \ldots, G_k$. Let us consider G_x for a fixed $x \in \{1, \ldots, k\}$. Let $r := \lceil \frac{n}{x} \rceil$. We have that $r \geq \frac{n}{x} \geq \frac{n}{k}$, so $\frac{kr}{n} \geq 1$. Therefore, by Lemma 12, the number of hyperedges crossing C in G_x is at least a quarter of the hyperedges of G_x. We also know that even if all hyperedges in G_x cross C, the weight of those hyperedges is only $(a_x - a_{x+1})(x/8)$. Therefore, the weight of hyperedges crossing C in G_x is between $(a_x - a_{x+1})(x/32)$ and $(a_x - a_{x+1})(x/8)$.

Next, we compute the weight of hyperedges crossing C in $G_{k+1}, \ldots, G_{\lceil n/2 \rceil - 1}$. Let us consider G_x for a fixed $x \in \{k+1, \ldots, \lceil \frac{n}{2} \rceil - 1\}$. Let $r := \lceil \frac{n}{x} \rceil$. Then, $2 \leq r \leq \frac{2n}{x} < \frac{2n}{k}$. Therefore, $\frac{kr}{n} \leq 2$, and hence,

$$\frac{k}{2x} \leq \frac{kr}{2n} \leq \min\left(\frac{kr}{n}, 1\right) \leq \frac{kr}{n} \leq \frac{2k}{x}.$$

From these inequalities and Lemma 12, we conclude that the number of hyperedges crossing C in G_x is at least a $\frac{k}{8x}$ fraction and at most a $\frac{8k}{x}$ fraction of hyperedges of G_x. Therefore, the weight of hyperedges crossing C in G_x is at least $(a_x - a_{x+1})k/64$ and at most $(a_x - a_{x+1})k$.

Therefore, if $d(C)$ is the weight of hyperedges crossing C in H, then

$$\frac{1}{64}\left(h(0) + a_{\lceil n/2 \rceil}k + \sum_{x=1}^{k}(a_x - a_{x+1})x + \sum_{x=k+1}^{\lceil n/2 \rceil - 1}(a_x - a_{x+1})k\right) \tag{4}$$

$$\leq \frac{1}{64}\left(h(0) + a_{\lceil n/2 \rceil}k + \sum_{x=1}^{k}2(a_x - a_{x+1})x + \sum_{x=k+1}^{\lceil n/2 \rceil - 1}(a_x - a_{x+1})k\right) \tag{5}$$

$$\leq d(C) \tag{6}$$

$$\leq \frac{h(0)}{64} + a_{\lceil n/2 \rceil}\frac{k}{4} + \sum_{x=1}^{k}(a_x - a_{x+1})\frac{x}{8} + \sum_{x=k+1}^{\lceil n/2 \rceil - 1}(a_x - a_{x+1})k \tag{7}$$

$$\leq h(0) + a_{\lceil n/2 \rceil}k + \sum_{x=1}^{k}(a_x - a_{x+1})x + \sum_{x=k+1}^{\lceil n/2 \rceil - 1}(a_x - a_{x+1})k. \tag{8}$$

Here, expression (8) is 64 times expression (4), so our proof is complete if we can show that expression (8) evaluates to $h(k)$ (recall that $h(k) = f(C)$. The next claim completes the proof by showing this. ◀

▷ Claim 14. For every $k \in \{0, 1, 2, \ldots, \lceil n/2 \rceil\}$, we have that

$$h(k) = h(0) + a_{\lceil n/2 \rceil} k + \sum_{x=1}^{k} (a_x - a_{x+1})x + \sum_{x=k+1}^{\lceil n/2 \rceil - 1} (a_x - a_{x+1})k$$

Proof. To show this, we simplify the two summations appearing on the RHS. The second summation telescopes to yield $(a_{k+1} - a_{\lceil n/2 \rceil})k$. To simplify the first summation, we note that $\sum_{x=1}^{j}(a_x - a_{x+1})x = \sum_{x=1}^{j}(2h(x) - h(x+1) - h(x-1))x$. For every x from 1 to $j-1$, $h(x)$ is added $2x$ times and subtracted $2x$ times in this summation, so it does not contribute at all. Therefore, we conclude that $\sum_{x=1}^{j}(a_x - a_{x+1})x = (j+1)h(x) - jh(x+1) - h(0)$.

Using the simplifications we have derived for each of the summations on the RHS, we find that the RHS is

$$\begin{aligned}
&h(0) + a_{\lceil n/2 \rceil} k + ((k+1)h(k) - kh(k+1) - h(0)) + (a_{k+1} - a_{\lceil n/2 \rceil})k \\
&= ka_{k+1} + (k+1)h(k) - kh(k+1) \\
&= k(h(k+1) - h(k)) + (k+1)h(k) - kh(k+1) \\
&= h(k).
\end{aligned}$$

◀

3.1.2 Symmetrized Concave Linear Functions

In this section, we prove Theorem 4—i.e., symmetrized concave linear functions are constant-hypergraph-approximable. Our approach is to first construct a hypergraph on a much larger vertex set than the ground set V, using the result of Theorem 13, and then contract subsets of the vertices of this hypergraph to obtain a hypergraph on the vertex set V with the desired property.

▶ **Theorem 15.** *Let V be a ground set, $w\colon V \to \mathbb{R}_+$, and $h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an increasing concave function. Then, the symmetric submodular function $f\colon 2^V \to \mathbb{R}_+$ defined by*

$$f(S) := h\left(\sum_{v \in S} w(v)\right) + h\left(\sum_{v \in V \setminus S} w(v)\right) - h\left(\sum_{v \in V} w(v)\right) - h(0)\ \forall\ S \subseteq V$$

is 128-hypergraph-approximable.

Proof. Let $n := |V|$. For ease of notation, we will use $w(S) := \sum_{v \in S} w(v)$ for all $S \subseteq V$. Let $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined by $g(x) := h(x) + h(w(V) - x) - h(w(V)) - h(0)$ for all $x \geq 0$. Then $f(S) = g(w(S))$. Since h is concave, and $h(w(V) - x)$ is $h(x)$ reflected over a vertical line at $x = w(V)/2$, the function $h(w(V) - x)$ is also concave. We also note that $-h(w(V)) - h(0)$ is a constant, and constant functions are concave. Therefore, g is a sum of concave functions, and hence, g is concave as well. Since g is concave, it is also continuous. Therefore, for every $x \in \mathbb{R}_+$ such that $g(x) \neq 0$, there exists a positive real number ε_x such that for every real number y with $x - \varepsilon_x < y < x + \varepsilon_x$, we have $g(x)/\sqrt{2} \leq g(y) \leq \sqrt{2}g(x)$. Let $\varepsilon_{min} = \min\{\varepsilon_{w(S)} \colon \emptyset \neq S \subset V\}$. Let $q := \lceil 2nw(V)/\varepsilon_{min} \rceil$. We note that $w(V)/q \leq \varepsilon_{w(S)}/2n$ for every $S \subset V$.

▷ Claim 16. There exist positive integers p_v for each $v \in V$ such that:

1. For every $v \in V$, we have that $w(v) - \frac{\varepsilon_{min}}{n} < \frac{p_v w(V)}{q} < w(v) + \frac{\varepsilon_{min}}{n}$.
2. $\sum_{v \in V} p_v = q$.

Proof. By our choice of q, for every $v \in V$, we have that $w(v) - w(V)/q > w(v) - \frac{\varepsilon_{min}}{n}$. Therefore, for each $v \in V$ we can choose a positive integer p_v such that $w(v) - \frac{\varepsilon_{min}}{n} < p_v w(V)/q \le w(v)$, and thus we can choose a collection of integers p_v which satisfies the first condition of the claim as well as $\sum_{v \in V} p_v \le q$.

Consider a collection of positive integers p_v for each $v \in V$ which maximizes $\sum_{v \in V} p_v$ subject to satisfying the first condition of the claim and the inequality $\sum_{v \in V} p_v \le q$. Suppose for contradiction that these integers do not satisfy the second condition of the claim. Then, $\sum_{v \in V} p_v < q$, so $\sum_{v \in V} p_v w(V)/q < w(V)$, so there must exist some $u \in V$ for which $p_u w(V)/q < w(u)$. By our choice of q, we have that

$$\frac{(p_u + 1)w(V)}{q} = \frac{p_u w(V)}{q} + \frac{w(V)}{q} < w(u) + \frac{w(V)}{q} \le w(u) + \frac{\varepsilon_{min}}{2n} < w(u) + \frac{\varepsilon_{min}}{n}.$$

Thus, we can increase p_u by 1 while still satisfying the first condition of the claim. Also, since $\sum_{v \in V} p_v < q$, we have that $1 + \sum_{v \in V} p_v \le q$, so we can increase p_u by 1 while maintaining that the sum of all the integers p_v is at most q. This contradicts our assumption that the integers p_v maximized $\sum_{v \in V} p_v$ subject to satisfying the first constraint of the claim and the inequality $\sum_{v \in V} p_v \le q$. Thus, a collection of positive integers satisfying the conditions of the claim exists. ◀

Choose a positive integer p_v for each $v \in V$ such that the chosen integers satisfy the conditions of Claim 16. For each $v \in V$, we create a set U_v containing p_v new vertices, and we define $U := \bigcup_{v \in V} U_v$. We note that $|U| = \sum_{v \in V} p_v = q$. We define functions $h_1 \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, $f_1 \colon [0, q] \to \mathbb{R}_+$, and $f_2 \colon [0, q] \to \mathbb{R}_+$ by $h_1(x) = h(xw(V)/q)$, $f_1(x) = h_1(x) + h_1(q - x) - h_1(q) - h_1(0)$, and $f_2(x) = f_1(x)/\sqrt{2}$. We note that h_1 is concave since it is a rescaling of h by a constant factor, and $h_1(q - x)$ is concave, since it is h_1 reflected over the vertical line at $x = q/2$. Thus, f_1 is the sum of concave functions, and hence, f_1 is concave. Finally, f_2 is a constant multiple of a concave function, so f_2 is concave as well. Furthermore, by definition, $f_2(q-x) = f_2(x)$. Applying Theorem 13 to q and f_2, we conclude that there exists a hypergraph H' with vertex set U whose cut function d' satisfies

$$d'(S) \le f_1(|S|)/\sqrt{2} \le 64 d'(S) \ \forall \ S \subseteq U,$$

Let H be the hypergraph obtained from H' by contracting each set U_v of vertices into a vertex $v \in V$. Let d be the cut function of H. To complete the proof, we will show that $d(S) \le f(S) \le 128 d(S)$ for every $S \subseteq V$. We first consider the special cases of $S = \emptyset$ and $S = V$. For both these cases, we have that $f(S) = 0 = d(S)$ by definition. Next, let us consider an arbitrary non-empty set $S \subset V$. Let $U_S := \bigcup_{v \in S} U_v$ be the corresponding set of vertices in H'. We note that by construction of H, we have that $d(S) = d'(U_S)$. Therefore,

$$d(S) \le f_1(|U_S|)/\sqrt{2} \le 64 d(S). \tag{9}$$

We note that

$$|U_S| = \sum_{v \in S} |U_v| = \sum_{v \in S} p_v.$$

Therefore, by definition,

$$f_1(|U_S|) = f_1\left(\sum_{v\in S} p_v\right)$$
$$= h_1\left(\sum_{v\in S} p_v\right) + h_1\left(q - \sum_{v\in S} p_v\right) - h_1(q) - h_1(0)$$
$$= h\left(\sum_{v\in S} \frac{p_v w(V)}{q}\right) + h\left(w(V) - \sum_{v\in S} \frac{p_v w(V)}{q}\right) - h(w(V)) - h(0)$$
$$= g\left(\sum_{v\in S} \frac{p_v w(V)}{q}\right).$$

For each $v \in S$, we have that $w(v) - \varepsilon_{min}/n < p_v w(V)/q < w(v) + \varepsilon_{min}/n$. We also have that $|S| \leq n$. Therefore,

$$w(S) - \varepsilon_{w(S)} \leq w(S) - \varepsilon_{min}$$
$$\leq w(S) - \frac{|S|\varepsilon_{min}}{n}$$
$$< \sum_{v\in S} \frac{p_v w(V)}{q}$$
$$< w(S) + \frac{|S|\varepsilon_{min}}{n}$$
$$\leq w(S) + \varepsilon_{min}$$
$$\leq w(S) + \varepsilon_{w(S)}.$$

So by definition of $\varepsilon_{w(S)}$, we have that

$$\frac{f(S)}{\sqrt{2}} = \frac{g(w(S))}{\sqrt{2}} \leq g\left(\sum_{v\in S} \frac{p_v w(V)}{q}\right) \leq \sqrt{2}g(w(S)) = \sqrt{2}f(S).$$

Thus $f(S)/\sqrt{2} \leq f_1(|U_S|) \leq \sqrt{2}f(S)$, and so by inequality (9) we have that

$$d(S) \leq f_1(|U_S|)/\sqrt{2} \leq f(S) \leq \sqrt{2}f_1(|U_S|) \leq 128d(S).$$

◀

3.2 Symmetrized Matroid Rank Functions

In this section, we prove Theorem 5 which states that symmetrized rank function of uniform and partition matroids are constant-hypergraph-approximable (see Section 1.2 for definitions of uniform and partition matroids). We begin with uniform matroids.

▶ **Lemma 17.** *The symmetrized rank function of a uniform matroid is* 64*-hypergraph-approximable.*

Proof. Let $r : 2^V \to \mathbb{R}_{\geq 0}$ be the rank function of the uniform matroid on ground set V with budget k and $r_{\text{sym}} : 2^V \to \mathbb{R}_{\geq 0}$ be the symmetrized rank function. We note that $r(S) = \min\{|S|, k\}$ for every $S \subseteq V$. If $k > |V|$, then $r_{\text{sym}}(S) = 0$ for every $S \subseteq V$ and hence, r_{sym} is 1-hypergraph-approximable using the empty hypergraph. So, we may assume that $k \leq |V|$. Then, for every $S \subseteq V$, we have that

$$\begin{aligned} r_{\text{sym}}(S) &= r(S) + r(V \setminus S) - r(V) \\ &= \min\{|S|, k\} + \min\{|V \setminus S|, k\} - \min\{|V|, k\} \\ &= \min\{|S|, |V \setminus S|, k, |V| - k\}. \end{aligned}$$

Let $n := |V|$ and consider the function $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$h(x) = \min\{x, n - x, k, n - k\}.$$

Then, h is concave on $[0, n]$ and $h(x) = h(n - x)$ for every $x \in [0, n]$ and $r_{\text{sym}}(S) = h(|S|)$ for every $S \subseteq V$. Therefore, by Theorem 13, we have that r_{sym} is 64-hypergraph-approximable. ◀

Next, we show that symmetrized rank functions of partition matroids are constant-hypergraph-approximable.

▶ **Theorem 18.** *The symmetrized rank function of a partition matroid is* 64*-hypergraph-approximable.*

Proof. Let $\mathcal{M} = (V, \mathcal{I})$ be a partition matroid on ground set V with rank function $r : 2^V \to \mathbb{Z}_{\geq 0}$ that is associated with the partition $V_1, \ldots, V_t$ of the ground set V and budgets $b_1, \ldots, b_t \in \mathbb{Z}_{\geq 0}$. For $i \in [t]$, we define a function $f_i : 2^{V_i} \to \mathbb{Z}_{\geq 0}$ by $f_i(S) := r_i(S) + r_i(V_i \setminus S) - r_i(V_i)$ where r_i is the rank function of the uniform matroid on ground set V_i with budget b_i. Then, the symmetrized rank function of the partition matroid $\mathcal{M}$ can be written as $r_{\text{sym}}(S) = \sum_{i=1}^{t} f_i(S \cap V_i)$. Moreover, each f_i is the symmetrized rank function of a uniform matroid. By Lemma 17, for each $i \in [t]$, there exists a weighted hypergraph G_i with cut function d_i such that

$$d_i(S) \leq f_i(S) \leq 64 d_i(S) \ \forall\ S \subseteq P_i.$$

Let G be the hypergraph on V formed by taking the union of the hypergraphs G_i for each $i \in [t]$. Since the vertex sets of the hypergraphs G_i are pairwise disjoint, the cut function $d : 2^V \to \mathbb{R}_{\geq 0}$ of G satisfies $d(S) = \sum_{i=1}^{t} d_i(S \cap V_i)$, and therefore G is a weighted hypergraph which fulfills the requirements of the theorem. ◀

4 Conclusion

In this work, we investigated the approximability of symmetric submodular functions using hypergraph cut functions. We proved that it suffices to understand the approximability of symmetrized matroid rank functions. On the upper bound side, we showed that symmetrized concave linear functions and symmetrized rank functions of uniform and partition matroids are constant-approximable using hypergraph cut functions. Our upper bounds for uniform and partition matroids raise the question of whether symmetrized rank functions of constant-depth laminar matroids are constant-approximable using hypergraph cut functions. On the lower bound side, we showed that there exist symmetrized matroid rank functions on n-element ground sets that cannot be $o(n^{1/3}/\log^2 n)$-approximated using hypergraph cut functions, thus ruling out constant-approximability of symmetric submodular functions using hypergraph cut functions. Our results raise the natural open question of whether every symmetric submodular function on n-element ground set is $O(\sqrt{n})$-hypergraph approximable.

Our strong lower bound also raises the question of whether we could trade off approximability against the number of vertices in the hypergraph. In particular, for every symmetric submodular function $f : 2^V \to \mathbb{R}_+$ defined over a n-element ground set V, does there exist a hypergraph over a vertex set $V' \supseteq V$ with cut function $d : 2^{V'} \to \mathbb{R}_{\geq 0}$ such that $d(A) \leq f(A) \leq \alpha d(A)$ for every $A \subseteq V$, where $\alpha = O(1)$ and $|V'| = O(2^n)$?

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A Proof of Lemma 12

We first show a few combinatorial inequalities that will be useful for our proof.

▷ Claim 19. For every integer $n \geq 2$, $k \in \{1, \ldots, \frac{n}{2}\}$, and $r \in \{2, \ldots, n-k\}$, we have that

1. $\left(1 - \frac{r}{n-k}\right)^k \leq \frac{\binom{n-k}{r}}{\binom{n}{r}} \leq \left(1 - \frac{r}{n}\right)^k$.
2. $\left(1 - \frac{k}{n-r}\right)^r \leq \frac{\binom{n-k}{r}}{\binom{n}{r}}$

Proof. **1.** We note that

$$\begin{aligned}\frac{\binom{n-k}{r}}{\binom{n}{r}} &= \frac{(n-k)!/(r!(n-k-r!))}{n!/(r!(n-r)!)} \\ &= \frac{(n-k)!}{n!} \cdot \frac{(n-r)!}{(n-k-r)!} \\ &= \prod_{i=0}^{k-1} \frac{n-r-i}{n-i}.\end{aligned}$$

We get the upper bound on $\binom{n-k}{r}/\binom{n}{r}$ by upper bounding every element of this product with $\frac{n-r}{n}$, and the lower bound by lower bounding every term of the product with $\frac{n-k-r}{n-k}$.

2. We note that

$$\begin{aligned}\frac{\binom{n-k}{r}}{\binom{n}{r}} &= \frac{(n-k)!/(r!(n-k-r!))}{n!/(r!(n-r)!)}\\ &= \frac{(n-k)!}{(n-k-r)!}\cdot\frac{(n-r)!}{n!}\\ &= \prod_{i=0}^{r-1}\frac{n-k-i}{n-i}.\end{aligned}$$

We obtain the lower bound by lower bounding every term of the product with $\frac{n-k-r}{n-r}$. ◀

▷ Claim 20. For every integer $n \geq 2$, $k \in \{1,\ldots,\frac{n}{2}\}$, and $r \in \{2,\ldots,n\}$, we have that

$$\frac{\binom{k}{r}}{\binom{n}{r}} \leq \left(\frac{k}{n}\right)^r.$$

Proof. If $k < r$, the bound trivially holds, because $\binom{k}{r} = 0$. Otherwise, we have

$$\begin{aligned}\frac{\binom{k}{r}}{\binom{n}{r}} &= \frac{k!/(r!(k-r)!)}{n!/(r!(n-r)!)}\\ &= \frac{k!}{(k-r)!}\cdot\frac{(n-r)!}{n!}\\ &= \prod_{i=0}^{r-1}\frac{k-i}{n-i}.\end{aligned}$$

Upper bounding every term in the product with $\frac{k}{n}$ gives the desired bound. ◀

We now restate and prove Lemma 12.

▶ **Lemma 12.** *For every integer $n \geq 2$, $r \in \{2,\ldots,n\}$, and $X \subseteq [n]$ with $1 \leq |X| \leq \frac{n}{2}$, the set of hyperedges $\delta(X)$ that cross X in a complete r-uniform n-vertex hypergraph has the following size bound:*

$$\frac{1}{4}\min\left\{\frac{|X|r}{n},1\right\} \leq \frac{|\delta(X)|}{\binom{n}{r}} \leq 4\min\left\{\frac{|X|r}{n},1\right\}.$$

Proof. Let $k := |X|$. We note that the hyperedges which cross X are exactly those which are neither fully contained in X, nor fully contained in $V \setminus X$. Thus, the number of rank r hyperedges in $\delta(X)$ is exactly $\binom{n}{r} - \binom{n-k}{r} - \binom{k}{r}$.

Suppose $r > n-k$. Then, since $k \leq n/2$, we have that $r > k$ as well, so $|\delta(X)| = \binom{n}{r} - \binom{n-k}{r} - \binom{k}{r} = \binom{n}{r}$, and so we have $\frac{|\delta(X)|}{\binom{n}{r}} = 1$. Thus, we immediately have that $\frac{1}{4}\min\left\{\frac{|X|r}{n},1\right\} \leq \frac{|\delta(X)|}{\binom{n}{r}}$. Furthermore, we have that $kr > k(n-k) = kn - k^2$, so $\frac{kr}{n} > \frac{kn-k^2}{n} = k - \frac{k^2}{n} \geq k - \frac{k}{2} = \frac{k}{2}$, so we have that $\frac{|\delta(X)|}{\binom{n}{r}} \leq 4\min\left\{\frac{|X|r}{n},1\right\}$. Henceforth we assume $r \leq n-k$.

We case on the value of k.

- Case 1: $k \geq n/r$. Then $\min\left\{\frac{|X|r}{n}, 1\right\} = 1$. Since $\frac{|\delta(X)|}{\binom{n}{r}}$ is the fraction of the hyperedges which are in $\delta(X)$, it is trivially upper bounded by 1, and thus by $4\min\left\{\frac{kr}{n}, 1\right\}$. Therefore, it remains to show the lower bound. We have that

$$\begin{aligned}\frac{|\delta(X)|}{\binom{n}{r}} &= \frac{\binom{n}{r} - \binom{n-k}{r} - \binom{k}{r}}{\binom{n}{r}} \\ &\geq 1 - \left(1 - \frac{r}{n}\right)^k - \left(\frac{k}{n}\right)^r \\ &\geq 1 - e^{-kr/n} - \left(\frac{k}{n}\right)^r \\ &\geq 1 - \frac{1}{e} - \frac{1}{4} \\ &\geq \frac{1}{4} \\ &= 0.25\min\left\{\frac{kr}{n}, 1\right\}.\end{aligned}$$

Here the second line follows from the upper bound in the first conclusion of Claim 19 and the upper bound in Claim 20, and the fourth follows from our assumptions that $n/r \leq k \leq n/2$ and $r \geq 2$.

- Case 2: $k < n/r$. Then $\min\left\{\frac{|X|r}{n}, 1\right\} = \frac{kr}{n}$. Once again, we need to show a lower bound and an upper bound. We begin with the lower bound:

$$\begin{aligned}\frac{|\delta(X)|}{\binom{n}{r}} &= \frac{\binom{n}{r} - \binom{n-k}{r} - \binom{k}{r}}{\binom{n}{r}} \\ &\geq 1 - \left(1 - \frac{r}{n}\right)^k - \left(\frac{k}{n}\right)^r \\ &\geq 1 - e^{-kr/n} - \left(\frac{k}{n}\right)^r \\ &\geq 1 - \left(1 - \frac{kr}{2n}\right) - \left(\frac{k}{n}\right)^r \\ &= \frac{kr}{2n} - \left(\frac{k}{n}\right)^r \\ &\geq \frac{kr}{2n} - \left(\frac{kr}{2n}\right)^2 \\ &= \frac{kr}{2n}\left(1 - \frac{kr}{2n}\right) \\ &\geq \frac{kr}{4n}.\end{aligned}$$

Here the second line follows from the upper bound in the first conclusion of Claim 19 and the upper bound in Claim 20, the fourth from the Taylor expansion of e^x, the sixth from the fact that $r \geq 2$, and the last line from the assumption that $k < n/r$.
Now we show the upper bound. Since the total number of hyperedges in the graph is $\binom{n}{r}$, we have that $|\delta(X)| \leq \binom{n}{r}$. Hence, $|\delta(X)|/\binom{n}{r} \leq 1$. It remains to show that $|\delta(X)|/\binom{n}{r} \leq 4|X|r/n = 4rk/n$. We consider 3 subcases based on the values of r and k:
  - Subcase 1: $r \geq n/4$. Since $r \geq n/4$ and $|X| \geq 1$, we have that $\frac{|X|r}{n} \geq \frac{1}{4}$. Therefore

$$\frac{|\delta(X)|}{\binom{n}{r}} \leq 1 \leq 4\frac{|X|r}{n}$$

- Subcase 2: $r < n/4$ and $k < r$. In this case, we have that $\binom{k}{r} = 0$. Therefore,

$$\begin{aligned}\frac{|\delta(X)|}{\binom{n}{r}} &= \frac{\binom{n}{r} - \binom{n-k}{r} - \binom{k}{r}}{\binom{n}{r}} \\ &= \frac{\binom{n}{r} - \binom{n-k}{r}}{\binom{n}{r}} \\ &= 1 - \frac{\binom{n-k}{r}}{\binom{n}{r}} \\ &\leq 1 - \left(1 - \frac{r}{n-k}\right)^k \\ &\leq 1 - e^{-2rk/(n-k)} \\ &\leq 1 - \left(1 - \frac{2rk}{n-k}\right) \\ &= \frac{2rk}{n-k} \\ &\leq \frac{4rk}{n}.\end{aligned}$$

The fourth line follows from the lower bound in the first conclusion of Claim 19. The fifth line follows from observing that $0 < k/(n-r) \leq 2/3$ (since $k \leq n/2$ and $r \leq n/4$) and $\ln(1-x) \geq -2x$ for every $x \in (0, 2/3]$. The sixth line follows from the Taylor expansion of e^x, and the last line follows from the fact that $k \leq n/2$.

- Subcase 3: $r < n/4$ and $k \geq r$. In this case we have that

$$\begin{aligned}\frac{|\delta(X)|}{\binom{n}{r}} &= \frac{\binom{n}{r} - \binom{n-k}{r} - \binom{k}{r}}{\binom{n}{r}} \\ &\leq \frac{\binom{n}{r} - \binom{n-k}{r}}{\binom{n}{r}} \\ &= 1 - \frac{\binom{n-k}{r}}{\binom{n}{r}} \\ &\leq 1 - \left(1 - \frac{k}{n-r}\right)^r \\ &\leq 1 - e^{-2rk/(n-r)} \\ &\leq 1 - \left(1 - \frac{2rk}{n-r}\right) \\ &= \frac{2rk}{n-r} \\ &\leq \frac{2rk}{3n/4} \\ &\leq \frac{4rk}{n}.\end{aligned}$$

The fourth line follows from the lower bound in the second conclusion of Claim 19. The fifth line follows from observing that $0 < k/(n-r) \leq 2/3$ (since $k \leq n/2$ and $r \leq n/4$) and $\ln(1-x) \geq -2x$ for every $x \in (0, 2/3]$. The sixth line follows from the Taylor expansion of e^x. The second to last line follows from the fact that $r < n/4$.

◀