THEORY OF COMPUTING
www.theoryofcomputing.org

Algorithms for Intersection Graphs of Multiple Intervals and Pseudo-Disks

Chandra Chekuri* Tanmay Inamdar†

April 14, 2021

Abstract: Intersection graphs of planar geometric objects such as intervals, disks, rectangles and pseudo-disks are well studied. Motivated by various applications, Butman et al. [9] in SODA 2007 considered algorithmic questions in intersection graphs of t-intervals. A t-interval is a union of at most t distinct intervals (here t is a parameter) — these graphs are referred to as Multiple-Interval Graphs. Subsequent work by Kammer et al. [22] in Approx-Random 2010 also considered t-disks and other geometric shapes. In this paper we revisit some of these algorithmic questions via more recent developments in computational geometry. For the minimum weight dominating set problem, we give a simple $O(t \log t)$ approximation for Multiple-Interval Graphs, improving on the previously known bound of t^2 . We also show that it is NP-hard to obtain an $o(t)$-approximation in this case. In fact, our results hold for the intersection graph of a set of t-pseudo-disks which is a much larger class. We obtain an $\Omega(1/t)$-approximation for the maximum weight independent set in the intersection graph of t-pseudo-disks. Our results are based on simple reductions to existing algorithms by appropriately bounding the union complexity of the objects under consideration.

*Chandra Chekuri is supported in part by NSF grants CCF-1526799 and CCF-1910149.
†Tanmay Inamdar is supported in part by NSF grant CCF-1615845.

ACM Classification: F.5.2.2, F.6.2

AMS Classification: 68W25

Key words and phrases: multiple interval graphs, geometric set cover, dominating set, independent set

1 Introduction

A number of interesting optimization problems can be modeled as packing and covering problems involving geometric objects in the plane such as intervals, disks, rectangles, triangles, convex objects and pseudo-disks. The intersection graphs of geometric objects are also of much interest for both theoretical and practical reasons. For instance the famous Koebe-Andreev-Thurston theorem shows that every planar graph can be represented as the intersection graph of interior-disjoint disks in the plane [28]. Interval graphs are another well-studied class of geometric intersection graph. Such a graph is induced by a finite collection of intervals on the real line — each vertex of the graph represents a (closed) interval on the real line, and there is an edge between two vertices iff the corresponding intervals have a non-empty intersection. Several algorithmic problems on interval graphs are motivated by practical applications such as scheduling and resource allocation.

Several papers in the past [7, 8, 9, 22] studied generalization of geometric intersection graphs to the setting where each (meta) object is now the union of a collection of base geometric objects. To make the discussion concrete we first discuss t-interval graphs. For an integer parameter $t \geq 1$, a t-interval is the union of at most t intervals. A t-interval graph is the intersection graph of a collection of t-intervals. These graphs are also called multiple-interval graphs. They have been well-studied from graph theoretic and algorithmic points of view. For instance, every graph with maximum degree Δ can be represented as a t-interval graph for $t = \lceil (\Delta+1)/2 \rceil$ [18]. This demonstrates the modeling power obtained by considering unions of simple geometric objects. Butman et al. [9], building on [7, 8] (which primarily studied the maximum independent set problem), considered several optimization problems over multiple-interval graphs such as minimum vertex cover, minimum dominating set and maximum clique. Unlike the case of interval graphs where these problems are tractable, the corresponding problems in multiple-interval graphs are NP-hard even for small values of t. Butmal et al. describe approximation algorithms for these problems and the approximation ratios depend on t. We refer the reader to [9] and references therein for a detailed discussion of the literature and applications of multiple-interval graphs. In a subsequent work, Kammer et al. [22, 21] studied (among other results) intersection graphs of t-disks (a t-disk is a union of at most t disks) and t-fat objects [1]; they obtained approximation algorithms for optimization problems on these graphs such as independent set, vertex cover and dominating set (see also [30]).

We now formally define the three problems that are central to this work. Let $G = (V,E)$ be an undirected graph with non-negative weights on vertices. $S \subseteq V$ is said to be a *dominating set* if for any vertex $u \in V$, either $u \in S$ or u has a neighbor in S. The Minimum Weight Dominating Set (MWDS) problem asks us to find a dominating set of the minimum weight. A subset of vertices, $I \subseteq V$ is said to be an *independent set* (or a stable set in some settings) if no distinct $u, v \in I$ are adjacent in G. The Maximum Weight Indpendent Set (MWIS) problem asks us to find an independent set of the maximum weight. Consider a set system $(X, \mathcal{S})$, where X is a finite ground set, and $\mathcal{S}$ is a collection of subsets of X. Each set in $\mathcal{S}$ has a non-negative weight. The Minimum Weight Set Cover (MWSC) problem asks to find a collection $\mathcal{S}' \subseteq \mathcal{S}$ of the minimum weight, such that $X = \bigcup_{S_i \in \mathcal{S}'} S_i$. We denote the unweighted versions (that is, all weights are unit) of MWDS, MWIS, and MWSC problems by MDS, MIS, and MSC,

[1]There are several equivalent definitions of *fatness* in the plane. We say that a geometric object O is α-fat for some $\alpha \geq 1$, if the ratio of the radius of the smallest enclosing disk of O to the radius of the largest disk enclosed by O, is upper bounded by a constant. We say that $\mathcal{R}$ is a collection of fat objects, if there exists a constant α such that every object in $\mathcal{R}$ is α-fat.

respectively.

Motivation and our contribution: In this paper we utilize powerful techniques from computational geometry [10, 11, 29] to provide algorithmic results for t-interval graphs, t-disks and other geometric objects, in a unified fashion. For some problems we obtain substantially improved approximation bounds that are near-optimal. Our results extend to t-pseudo-disks[2] while techniques in earlier work that exploited properties of intervals, [9] or fatness properties of the underlying objects [22], do not apply.

Before stating our results in full generality, we consider the following geometric covering problem. Given a collection of points on the line, and a collection of weighted intervals, find the minimum weight subset of the given intervals that cover all the points. This is a special case of the MWSC, and can be solved efficiently via dynamic programming or via mathematical programming — the natural LP relaxation in the interval case happens to yield an an integer polytope since the incidence matrix between intervals and points is totally unimodular (TUM) (it has the consecutive ones property). Now consider the same problem where we need to cover a collection of points by (weighted) t-intervals. Approximation algorithms for this problem were considered by Hochbaum and Levin [19] (in the more general setting of multicover) — they derived a relatively straightforward t-approximation for this problem by reducing it, via the natural LP relaxation, to the case of $t = 1$. As far as we are aware, this was the best known approximation to this problem. A natural question is whether the bound of t can be improved. In this paper we show that an $O(\log t)$-approximation can be obtained via tools from computational geometry such as shallow-cell complexity and quasi-uniform sampling. It is an easy observation that a MWSC instance where each set has at most t elements can be reduced to covering points by t-intervals; thus, for large values of t one obtains an $\Omega(\log t)$ hardness under the assumption that $\mathsf{P} \neq \mathsf{NP}$. The geometric machinery allows us to derive an $O(\log t)$-approximation in the much more general setting of covering points by t-pseudo-disks.

We state our results for t-pseudo-disks which capture several shapes of interest including intervals and disks. The geometric approach applies in more generality but here we confine our attention to pseudo-disks.

- An $O(\log t)$ approximation for minimum weight cover of points by t-pseudo-disks, in other words, the instances of MWSC induced by points and t-pseudo-disks.
- An $O(t \log t)$ approximation for MWDS in t-pseudo-disk graphs. Even for t-intervals the best known previous approximation was t^2 [9]. We observe that, via a simple reduction from Hypergraph Vertex Cover, it is NP-hard to approximate MDS within a factor better than $(t-1-\varepsilon)$ for any fixed $\varepsilon > 0$ in t-interval graphs. Under the Unique Games Conjecture [23, 24], this lower bound can be slightly improved to $(t-\varepsilon)$.
- An $\Omega(1/t)$ approximation for MWIS in t-pseudo-disk graphs. A $1/(2t)$-approximation was known for t-intervals [8]. For more general shapes such as disks and fat objects, the approximation bounds in [21] depend on t and fatness parameters (see [21] for the actual definition). Our approach also works for packing weighted t-pseudo-disks into capacitated points.

[2] A formal definition is given later in the paper. Informally a collection of connected regions in the plane is called a collection of pseudo-disks if the boundaries of any two of the regions intersect in at most two points. Note that the property is defined by the entire collection.

We note that although our results are obtained via simple reductions to existing algorithms in geometric packing and covering, these algorithms use fairly sophisticated ideas. Consequently, in some cases, the constants in the approximation guarantees may be worse compared to the known results. However, our results are applicable to a much more general class of geometric objects. Furthermore, it may be possible to improve the constants for the special case of intervals.

We describe the necessary geometric background in the next subsection.

1.1 Background from Geometric Approximation via LP Relaxations

The approximability of MWSC, MWDS and MWIS in general graphs is well-understood with essentially tight upper and lower bounds known. In particular it is NP-hard to obtain constant approximations for MWDS and MWIS problems. However, in various geometric setting it is possible to obtain improved algorithms including approximation schemes (PTASes and QPTASes [11, 17, 10, 2, 26, 1]) and constant factor approximations [11, 10, 17]. In this paper we are interested in LP-based approximations for MWSC and MWIS that have been established via techniques that rely on *union complexity* of the underlying geometric objects. Union complexity measures the worst-case representation size of the union of a given collection of objects of a particular type or shape. In the setting of planar obejcts the typical measure is the number of vertices in the arrangement that appear on the boundary of the union. It is well-known that many geometric objects such as intervals (on a line), disks and squares (in the plane) have linear union complexity. In fact, this holds for an even larger class of geometric objects, namely pseudo-disks [3].

Bounds on union complexity have been used to obtain constant factor and sub-logarithmic approximations for geometric MWSC and its variants (see [14, 12, 29, 10] and [25] for a survey). Chan and Har-Peled [11] showed how union complexity can also be used to obtain improved approximations for the MWIS problem. They give an LP rounding algorithm with approximation guarantee $\Omega(n/u(n))$ for computing an MWIS of n objects with union complexity $u(\cdot)$. We use this result to give $\Omega(1/t)$-approximation for computing MWIS of t-pseudo-disks. We note that although this implies $\Omega(1/t)$-approximation for MWIS of t-intervals, t-disks and t-squares; these results were already known [7, 22]. However, they use certain "fatness" properties of the underlying geometric objects, which do not extend to pseudo-disks.

Shallow-cell complexity (SCC) is a generalization of the notion of union complexity to abstract set systems [10]. Low SCC has been used to obtain improved approximations for MWSC and MWDS in geometric settings as well some combinatorial settings. Here, the general approach is to round a feasible LP solution using a framework called *Quasi-Uniform Sampling* introduced by Varadarajan [29] originally for geometric settings, and further refined and improved by Chan et al. [10]. We use this framework and the result in [10] to obtain an $O(t \log t)$-approximation for MWDS of t-pseudo-disks by appropriately bounding the SCC of a related instance via known results on MWDS of disks and pseudo-disks [17, 5]. This coupled with other ideas yields our results on MWSC and MWDS for t-pseudo-disks.

Orgnization. Section 2 introduces some relevant notation and definitions. Section 3 describes algorithms for covering problems MWDS and MWSC with a focus on the more involved MWDS problem. Section 4 describes algorithms for MWIS and a generalization.

2 Preliminaries

A t-object is defined as the union of at most t geometric objects, where t is a positive integer. Without loss of generality, we assume that each t-object is a union of exactly t objects. We are typically interested in the case when the base objects come from a specific class of geometric shapes, such as intervals, disks, and pseudo-disks.

Definition 2.1 (Pseudo-disks and t-pseudo-disks)**.** A family $\mathcal{S}$ of connected regions bounded by simple closed curves in general position in the plane is called a collection of *pseudo-disks*, if the boundaries of any distinct $S_i, S_j \in \mathcal{S}$ intersect at most twice. Let $\mathcal{S}$ be a collection of pseudo-disks, and let $\mathcal{S}'$ be a collection of objects obtained by taking union of at most t objects from $\mathcal{S}$. Then, we say that $\mathcal{S}'$ is a collection of t-pseudo-disks.

Consider a t-object O_i. We use $o_i^{(1)}, o_i^{(2)}, \dots, o_i^{(t)}$ to denote the objects whose union equals O_i. Then, by slightly abusing the notation, we also denote by O_i the set $\{o_i^{(1)}, o_i^{(2)}, \dots, o_i^{(t)}\}$. Note that, in some cases it may be desirable to require that the objects $o_i^{(1)}, o_i^{(2)}, \dots, o_i^{(t)}$ are pairwise disjoint. However we do not impose such a restriction on the objects and allow a more general model where they may intersect.

Next, we formally define the notion of union complexity, which is used to obtain the improved approximation for MWIS in Section 4.

Definition 2.2 (Union complexity)**.** Let $\mathcal{S}$ be a set of geometric objects in $\mathbb{R}^2$. We say that $\mathcal{S}$ has union complexity $u(\cdot)$ for a function $u : \mathbb{Z}_+ \to \mathbb{Z}_+$ if the following condition is true: for any subset of objects $\mathcal{R} \subseteq \mathcal{S}$, the number of vertices on the boundary of the union of objects in $\mathcal{R}$ is at most $u(|\mathcal{R}|)$.

In the rest of the paper we assume that $u(n) \geq n$ for any $n \geq 1$ and that u is non-decreasing. These are natural assumptions for the settings we consider.

3 Minimum Weight Dominating Set and Set Cover

As in Butman et al. [9], we consider the following slight generalization of the Minimum Dominating Set problem. Here, we are given a undirected bipartite intersection graph of t-objects. More formally, $G = (\mathcal{R} \sqcup \mathcal{B}, E)$, where $\mathcal{R} = \{R_1, \dots, R_N\}$ and $\mathcal{B} = \{B_1, \dots, B_M\}$ are sets of red and blue t-objects respectively. There is an edge between $R_i \in \mathcal{R}$ and $B_j \in \mathcal{B}$ if $R_i \cap B_j \neq \emptyset$. Since each vertex corresponds to a (red or blue) t-object, we will use the term 'vertex' and 't-object' interchangeably. Each $R_i \in \mathcal{R}$ has a non-negative weight w_i. The goal is to find a subset $\mathcal{R}' \subseteq \mathcal{R}$ of minimum weight, such that every blue vertex B_j has a neighbor in the set $\mathcal{R}'$. Note that the standard MWDS problem is equivalent to the setting where $\mathcal{R} = \mathcal{B}$. In the following, we consider the more general setting where $\mathcal{R}$ and $\mathcal{B}$ may be different.

3.1 LP Relaxation and Rounding

Let $\mathcal{I} = (\mathcal{R}, \mathcal{B})$ denote the given instance of the (generalized) MWDS. In an integer programming formulation we have a $\{0,1\}$ variable x_i corresponding to a red t-object R_i which is intended to be assigned 1 if R_i is selected in the solution, and is assigned 0 otherwise. We relax the integrality constraints and describe the natural LP relaxation below.

$$\begin{aligned} \text{minimize} \quad & \sum_{R_i \in \mathcal{R}} w_i x_i \\ \text{subject to} \quad & \sum_{R_i : B_j \cap R_i \neq \emptyset} x_i \geq 1, \quad \forall B_j \in \mathcal{B} \qquad (3.1) \\ & x_i \in [0,1], \qquad \forall R_i \in \mathcal{R} \qquad (3.2) \end{aligned}$$

Let x be an optimal LP solution for the given instance $\mathcal{I} = (\mathcal{R}, \mathcal{B})$. Since x is feasible, for every blue t-object B_j, we have $\sum_{R_i : B_j \cap R_i \neq \emptyset} x_i \geq 1$. For each $B_j \in \mathcal{B}$, let $b_j^{(k)} \in B_j$ denote an object maximizing the quantity $\sum_{R_i : b_j^{(k)} \cap R_i \neq \emptyset} x_i$, where the ties are broken arbitrarily. For simplicity, we denote $b_j^{(k)}$ by b'_j. Let $\mathcal{B}' := \{b'_j : B_j \in \mathcal{B}\}$. Note that for any $b'_j \in \mathcal{B}'$, $\sum_{R_i : b'_j \cap R_i \neq \emptyset} x_i \geq 1/t$.

For any $R_i \in \mathcal{R}$, let $x'_i = \min\{t x_i, 1\}$, and let x' denote the resulting solution. Then, for any $b'_j \in \mathcal{B}'$, $\sum_{R_i : b'_j \cap R_i \neq \emptyset} x'_i \geq 1$. We emphasize that $\mathcal{B}'$ is a collection of (1-)objects, whereas $\mathcal{R}$ is a collection of t-objects. The following observation follows from the definition of x'.

Observation 1. The solution x' is feasible for the instance $\mathcal{I}' = (\mathcal{R}, \mathcal{B}')$, and we have that $\sum_{R_i \in \mathcal{R}} w_i x'_i \leq t \cdot \sum_{R_i \in \mathcal{R}} w_i x_i$.

The preceding step to reduce to $\mathcal{I}'$ is essentially the same as in [9].

Shallow-Cell Complexity Now, we describe how to round x' to an integral solution via Quasi-Uniform Sampling technique. The standard LP relaxation for MWSC is as follows.

$$\begin{aligned} \min \sum_{S_i \in \mathcal{S}} w_i x_i & & & \\ \sum_{S_i \ni j} x_i & \geq & 1 & \quad \forall j \in X \\ x_i & \geq & 0 & \quad \forall S_i \in \mathcal{S} \end{aligned}$$

Let $A \in \{0,1\}^{M \times N}$ denote the constraint matrix in the LP relaxation above. Each row of A corresponds to an element, and each column corresponds to a set. The entry $A_{ij} = 1$ if the element j is contained in S_i, otherwise $A_{ij} = 0$.

The following crucial definition is from [10].

Definition 3.1 (Shallow-Cell Complexity)**.** Let $1 \leq k \leq n \leq N$. Let S be any set of n columns and let A_S be the matrix restricted to the columns of S. If the number of distinct rows in A_S with at most k ones is bounded by $f(n,k)$ for any choice of n, k and S, then the instance is said to have the shallow-cell complexity $f(n,k)$.

Using bounds on the Shallow-Cell Complexity of an MWSC instance, it is possible to round a feasible LP solution using a technique known as Quasi-Uniform Sampling [10, 29].

Theorem 3.2 ([10, 29])**.** *Consider an MWSC instance with Shallow-Cell Complexity $f(n,k) = n\phi(n)\cdot k^c$, where $\phi(n) = O(n)$, and $c \geq 0$ is a constant. Then, there exists an algorithm to round any feasible LP solution for this instance within a factor of $O(\max\{\log \phi(N), 1\})$, where the constant hidden in Big-Oh notation depends on the exponent c.*

Usually, $\phi(n)$ is a function of n such that $\phi(n) = O(n)$. However, the same guarantee holds even when ϕ is independent of n . When we apply this theorem, we will set $\phi(n) = O(t^4)$, which is independent of n.

Now, we consider an instance $\mathcal{I}' = (\mathcal{R}, \mathcal{B}')$ of the MWDS problem obtained in the previous section. Recall that $\mathcal{R}$ is a collection of t-objects and $\mathcal{B}$ is a collection of 1-objects. Now, let $\mathcal{R}' = \{r_i^{(k)} \in R_i : R_i \in \mathcal{R}\}$ denote the collection of constituent red 1-objects from $\mathcal{R}$. Let $\mathcal{I}'' = (\mathcal{R}', \mathcal{B}')$ denote the MWDS instance thus obtained. Notice that an MWDS instance can also be thought of as a MWSC instance. We first prove the following simple lemma.

Lemma 3.3. *Let $\mathcal{I}', \mathcal{I}''$ be MWDS instances as defined above. If the shallow-cell complexity of $\mathcal{I}''$ is $f(n,k)$, then the shallow-cell complexity of the corresponding instance $\mathcal{I}'$ is $g(n,k) \leq f(nt, kt)$ for any $1 \leq k \leq n \leq N$.*

Proof. We prove this fact from the definition of the shallow-cell complexity. Let $A^{\mathcal{I}'}$ denote the constraint matrix corresponding to the instance $\mathcal{I}'$. Fix some positive integers n,k such that $1 \leq k \leq n \leq |\mathcal{R}|$, and fix a set $S \subseteq \mathcal{R}$ of columns (i.e., t-objects), where $|S| = n$. Let $A_S^{\mathcal{I}'}$ denote the constraint matrix restricted to the columns corresponding to S. Let P denote the set of distinct rows (i.e., blue objects) in $A_S^{\mathcal{I}'}$ with at most k ones. We seek to bound $|P|$.

Let $S' = \{r_i^{(k)} \in R_i : R_i \in S\}$ be the corresponding constituent 1-objects. Note that $S' \subseteq \mathcal{R}'$, and $|S'| \leq nt$. Let $A^{\mathcal{I}''}$ denote the constraint matrix corresponding to the instance $\mathcal{I}''$ and let $A_{S'}^{\mathcal{I}''}$ denote $A^{\mathcal{I}''}$ restricted to the columns of S'. Let P' denote the set of rows in $A_{S'}^{\mathcal{I}''}$ with at most kt ones. Note that, if a row has at most k ones in $A_S^{\mathcal{I}'}$, then it corresponds to exactly one row in P'. Therefore, $|P| \leq |P'| \leq f(nt, kt)$, where the last inequality follows from the definition of the Shallow-Cell Complexity of the instance $\mathcal{I}''$. □

Now, we state the known bounds on the Shallow-Cell Complexity of an MWDS instance induced by red and blue collections of pseudo-disks.

Theorem 3.4 ([5])**.** *The Shallow-Cell Complexity of an MWDS instance induced by collections of pseudo-disks is at most $f(n,k) = O(nk^3)$.*

Combining Theorems 3.2 and 3.4 and Lemma 3.3, we obtain the following result.

Theorem 3.5. *There exists a randomized polynomial time $O(t \log t)$ approximation algorithm for the Red-Blue Dominating Set induced by a set of t-pseudo-disks.*

Proof. Let $\mathcal{I} = (\mathcal{R}, \mathcal{B})$ be the original instance and let x be an optimal LP solution. Define instance $\mathcal{I}' = (\mathcal{R}, \mathcal{B}')$ and the corresponding LP solution x' as before. From Observation 1, we have that $\sum_{R_i \in \mathcal{R}} w_i x_i' \leq t \cdot$

$\sum_{R_i \in \mathcal{R}} w_i x_i$. By Lemma 3.3 the shallow-cell complexity of $\mathcal{I}'$ is $g(n,k) \leq f(nt,kt)$, where $f(n,k) = O(nk^3)$ by Theorem 3.4. Therefore, $g(n,k) \leq O(nt^4k^3)$. Now, using the algorithm from Theorem 3.2 with $\phi(n) = O(t^4)$, we can round x' to an integral solution of cost at most $O(\log t) \cdot \sum_{R_i \in \mathcal{R}} w_i x'_i \leq O(t \log t) \cdot \sum_{R_i \in \mathcal{R}} w_i x_i$, where the inequality follows from Observation 1. □

Remark 3.6. Suppose we have an instance of $(\mathcal{R}, \mathcal{B})$ of generalized MWDS where each red object is a t_R-pseudo-disk and each blue object is a t_B-pseudo-disk. Then the preceding analysis can be extended to obtain an $O(t_B \log t_R)$ approximation. We note that for the analogous version of intervals, a $(t_B \cdot t_R)$-approximation was obtained in [9].

Geometric MWSC with t-objects. Consider a Geometric MWSC instance $(X, \mathcal{S})$, where X is a set of points, and $\mathcal{S}$ is a collection of N sets induced by t-objects. Recall that the goal is to find a minimum-weight collection of t-objects $\mathcal{S}'$ that *covers* the set of points X. Furthermore, suppose that the underlying geometric (1-)objects define a Set Cover instance with Shallow-Cell Complexity $f(n,k) = nk^c$, for constant c. This includes geometric objects such as intervals on a line, and (pseudo-)disks in the plane. Then, using similar arguments, one can bound the shallow-cell complexity of the MWSC instance by $O(nk^c t^{c+1})$. Then Theorem 3.2 implies an $O(\log t)$-approximation.

If the underlying geometric objects are a set of fat triangles, then the Shallow-Cell Complexity can be bounded by $f(n,k) = n \log^* n \cdot k^c$, for some constant c [4, 10]. Using similar arguments, we can bound the Shallow-Cell Complexity of the MWSC instance by $f(nt,kt) = n \log^*(nt) \cdot t^{c+1} k^c$. Then, Theorem 3.2 implies an $O(\log(t^{c+1} \cdot \log^*(Nt))) = O(\log t + \log\log^* Nt)$-approximation. Therefore, we obtain the following result.

Theorem 3.7. *There exists an $O(\log t)$ approximation algorithm for covering a set of points by t-objects, where the underlying geometric objects are intervals on a line, or (pseudo-)disks in the plane. If the underlying objects are fat triangles in the plane, we get a $O(\log t + \log\log^* Nt)$-approximation.*

We note that this result improves on t-approximation for covering points by t-intervals, which follows from the result of Hochbaum and Levin [19]. We also note that $O(\log t)$ is tight up to constant factors even for covering points by t-intervals, which follows from a reduction from the Set Cover problem as observed in [9].

3.2 Integrality of MWDS LP for Intervals

In this subsection, we prove that the MWDS LP for intervals is integral. Let $\mathcal{I} = (\mathcal{R}, \mathcal{B})$ be an MWDS instance, where $\mathcal{R}$ and $\mathcal{B}$ are collections of intervals on a line. Butman et al. [9] proved this using a primal-dual algorithm that constructs an integral solution of the same cost as the LP. We give a simpler proof of this fact by using structure of the constraint matrix.

First, we preprocess blue intervals such that no blue interval is completely contained inside another blue interval, and let $\mathcal{I}'$ denote the resulting instance. Suppose $B_1, B_2 \in \mathcal{B}$ are two intervals such that $B_1 \subset B_2$. Then, any feasible integral solution must include a red interval R that intersects B_1, and thus B_2. Furthermore, the LP constraint corresponding to B_2 is implied by the constraint corresponding to B_1. Therefore, the feasible regions of the LP's corresponding to $\mathcal{I}$ and $\mathcal{I}'$ are also the same.

Now, in the following theorem we show that the constraint matrix of the LP corresponding to $\mathcal{I}'$ satisfies the consecutive ones property, and thus it is totally unimodular. This implies that the LP corresponding to $\mathcal{I}_1$ (and therefore $\mathcal{I}$) is integral (see, e.g., [27]).

Theorem 3.8. *Let $\mathcal{I} = (\mathcal{R}, \mathcal{B})$ be an MWDS instance, where $\mathcal{R}$ and $\mathcal{B}$ are collections of (1-)intervals. Then, the MWDS LP is integral.*

Proof. Let us denote the left (resp. right) endpoint of an interval I by $\mathsf{Left}(I)$ (reps. $\mathsf{Right}(I)$). Let A denote the constraint matrix corresponding to the preprocessed instance, where the rows (i.e., blue intervals) are sorted in the non-decreasing order of their left endpoints. We show that A has the consecutive ones property for any column.

Consider any column (i.e., a red interval) R. Let B_j and B_k denote the first (leftmost) and last (rightmost) blue intervals intersecting R in this order respectively. Note that $j \le k$. Now, suppose for contradiction that there is an interval B_ℓ with $j < \ell < k$ that does not intersect R. Let us consider two cases.

If $\mathsf{Right}(B_\ell) < \mathsf{Left}(R)$, then we have that $\mathsf{Left}(B_j) < \mathsf{Left}(B_\ell) \le \mathsf{Right}(B_\ell) < \mathsf{Left}(R) \le \mathsf{Right}(B_j)$. Here, the third and fourth inequalities follow from the assumptions that $R \cap B_\ell = \emptyset$ and $R \cap B_j \ne \emptyset$ respectively. However, this implies that $B_\ell \subset B_j$, which is a contradiction.

On the other hand, if $\mathsf{Right}(B_\ell) > \mathsf{Left}(R)$, then it must be the case that $\mathsf{Left}(B_\ell) > \mathsf{Right}(R)$; otherwise $R \cap B_\ell \ne \emptyset$, which is a contradiction. However, this implies that $\mathsf{Left}(R) \le \mathsf{Right}(R) < \mathsf{Left}(B_\ell) \le \mathsf{Left}(B_k) \le \mathsf{Right}(B_k)$, where the third inequality follows from the fact that $\ell < k$. However, this contradicts the assumption that $B_k \cap R \ne \emptyset$.

Thus, the column corresponding to R satisfies the consecutive ones property. □

3.3 Hardness Results for MWDS

We show that it is NP-hard to obtain a $t - 1 - \varepsilon$ approximation for MWDS problem for t-intervals, for any $t \ge 3$ and $\varepsilon > 0$. The lower bound can be improved to $t - \varepsilon$ for any $t \ge 2$ and $\varepsilon > 0$, assuming the Unique Games Conjecture (UGC). In fact, this result follows even in the very restricted setting, where $\mathcal{R}$ is a collection of points, and $\mathcal{B}$ is a collection of t-points, i.e., a union of at most t distinct points. Note that this is a special case where $\mathcal{R}$ and $\mathcal{B}$ are collections of t-intervals.

Let $(X, \mathcal{S})$ be an f-uniform instance of MWSC. That is, any element in X is contained in exactly f sets of $\mathcal{S}$. We reduce this to the MWDS problem as follows. For every set $S_i \in \mathcal{S}$, we add a distinct red point R_i on a line, and set its weight equal to that of S_i. For any element $e_j \in X$, we add at most f blue points coinciding with points R_i, where $e_j \in S_i$. Note that a blue t-point is "covered" if we select a red point that coincides with its constituent t points. Thus, there is a one-to-one correspondence between a feasible solution to the original set cover instance, and a feasible solution to the MWDS instance. We have the following hardness results for the f-uniform MWSC problem.

Theorem 3.9. *It is NP-hard to obtain a $f - 1 - \varepsilon$ approximation for f-uniform Set Cover, for any $f \ge 3$ and $\varepsilon > 0$ [15].*
Assuming UGC, the lower bound can be improved to $f - \varepsilon$ for any $f \ge 2$ and $\varepsilon > 0$ [24].

Therefore, from the above reduction, we get the following hardness results for the MWDS problem.

Theorem 3.10. *It is* NP*-hard to obtain a* $t-1-\varepsilon$ *approximation for the special case of MWDS, where* $\mathcal{R}$ *is a set of points and* $\mathcal{B}$ *is a set of* t*-points.*
Assuming UGC, the lower bound can be improved to $t-\varepsilon$ *for any* $t \geq 2$ *and* $\varepsilon > 0$.

4 Maximum Weight Independent Set

We consider MWIS of t-objects. Here, we are given a set $\mathcal{S} = \{S_1, \dots, S_n\}$[3], where each $S_i \in \mathcal{S}$ is a t-object, and has a non-negative weight w_i. The goal is to find maximum weight independent set $\mathcal{S}' \subseteq \mathcal{S}$. That is, for any distinct $S_i, S_j \in \mathcal{S}'$, we must have $S_i \cap S_j = \emptyset$.

Let $\mathcal{V}(\mathcal{S})$ denote the set of vertices in the arrangement of $\mathcal{S}$ (see [3] for a formal definition). We use the LP rounding algorithm for Maximum-Weight Independent Set from [11]. First, we state the LP relaxation from [11].

$$\begin{aligned} \text{maximize} \quad & \sum_{S_i \in \mathcal{S}} w_i x_i \\ \text{subject to} \quad & \sum_{S_i \ni p} x_i \leq 1, \qquad \forall p \in \mathcal{V}(\mathcal{S}) \qquad (4.1) \\ & x_i \in [0,1], \qquad \forall S_i \in \mathcal{S} \qquad (4.2) \end{aligned}$$

We have a simple observation that relates the union complexities of t-objects and the corresponding (1-)objects.

Observation 2. Let $\mathcal{S}$ be a collection of t-objects, and suppose the collection of underlying (1-)objects has union complexity $u(\cdot)$. Then, the union complexity of any k objects in $\mathcal{S}$ is at most $u(kt)$.

Proof. Let $\mathcal{R} \subseteq \mathcal{S}$ be any subset of t-objects. Let $\mathcal{R}' = \{r_i^{(k)} \in R_i : R_i \in R\}$ denote the underlying set of (1-)objects. Note that $|R'| \leq t \cdot |R|$. Since the underlying collection of (1-)objects has union complexity $u(\cdot)$, the number of arcs on the boundary of R' is at most $u(|R'|) \leq u(t \cdot |R|)$. □

Chan and Har-Peled [11] give a randomized LP rounding algorithm with the following guarantee.

Theorem 4.1. *There is an* $\Omega(n/u(n))$*-approximation algorithm for an MWIS instance, for a collection of* n *geometric objects with union complexity* $u(\cdot)$.

Combining Observation 2 and Theorem 4.1, we get the following result.

Theorem 4.2. *There is an* $\Omega(n/u(nt))$*-approximation algorithm for an MWIS instance for an instance of* t*-objects, where the underlying* (1-)*objects have union complexity* $u(\cdot)$.

In particular, since pseudo-disks have linear union complexity [3], an $\Omega(1/t)$-approximation follows for MWIS of t-pseudo-disks. As noted in the introduction, this result is tight up to constant factors even for the case of t-intervals.

[3] In this section, we assume that we are given the geometric representation of the t-objects in $\mathcal{S}$.

Packing t-objects We consider a related problem called Maximum Weight Region Packing. We are given a collection of t-objects $\mathcal{S}$, and a set of points P. Each t-object S_i has a weight w_i and each point $p \in P$ has a capacity $c(p)$, which is a positive integer. The goal is to find a maximum-weight collection $\mathcal{S}'$ of t-objects, such that for each point $p \in P$, the number of regions of $\mathcal{S}'$ that contain p is at most $c(p)$. Note that MWIS is a special case when $P = \mathcal{V}(S)$ is the set of all points in arrangement and $c(p) = 1$ for each $p \in P$.

Extending the LP rounding algorithm of Chan and Har-Peled [11] for the MWIS problem, Ene et al. [16] gave an $\Omega((\frac{n}{u(n)})^{1/C})$-approximation for Maximum Weight Region Packing problem, where C is the minimum capacity of any point. Using similar arguments as in MWIS, we obtain the following result.

Theorem 4.3. *There exists a polynomial $\Omega(1/t^{1/C})$-approximation algorithm for Maximum Weight Region Packing with t-pseudo-disks.*

5 Concluding Remarks

In geometric settings, Quasi-Uniform Sampling has been used for Set Multicover [6], and other related ideas have been used for Partial Set Cover [20, 13]. Our results for MWDS and MWSC using t-objects can be extended to these settings. We briefly sketch the ideas for an interested reader.

The results in [20, 13] establish a generic high-level reduction: an α-approximation via the natural LP relaxation for MWSC can be translated into an $O(\alpha)$-approximation for Partial Set Cover for hereditary instances. The geometric instances considered here are hereditary, and hence our results for MWSC can be extended.

Set Multicover is a generalization of Set Cover. In the geometric setting of interest here, each point p has integer demand $d_p \geq 1$, and the goal is choose a minium weight subset of objects so that each point is contained in at least d_p objects. Bansal and Pruhs [6] extended the shallow-sell complexity based framework from Set Cover to Set Multicover. Since we prove our results by establishing shallow-cell complexity for t-objects, it is not hard to verify that the results in [6] also apply to Set Multicover with t-pseudo-disks.

Finally, we note that even for the special case of MWDS of t-intervals, there is a gap between $O(t \log t)$-approximation and the lower bound of $\Omega(t)$, as shown in Section 3. Resolving this gap is an interesting open question. Obtaining smaller leading constants in the approximation ratios, and avoiding the somewhat complicated machinery of union complexity, for special cases such as t-intervals, is also of interest.

References

[1] Anna Adamaszek, Sariel Har-Peled, and Andreas Wiese. Approximation schemes for independent set and sparse subsets of polygons. *Journal of the ACM (JACM)*, 66(4):1–40, 2019. 4

[2] Anna Adamaszek and Andreas Wiese. Approximation schemes for maximum weight independent set of rectangles. In *54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, 2013, Berkeley, CA, USA*, pages 400–409, 2013. 4

[3] Pankaj K Agarwal, János Pach, Micha Sharir, J Goodman, and R Pollack. State of the union (of geometric objects): A review. *Computational Geometry: Twenty Years Later. American Mathematical Society*, 2007. 4, 10

[4] Boris Aronov, Mark de Berg, Esther Ezra, and Micha Sharir. Improved bounds for the union of locally fat objects in the plane. *SIAM J. Comput.*, 43(2):543–572, 2014. 8

[5] Boris Aronov, Anirudh Donakonda, Esther Ezra, and Rom Pinchasi. On pseudo-disk hypergraphs. *Computational Geometry*, 92:101687, 2021. 4, 7

[6] Nikhil Bansal and Kirk Pruhs. Weighted geometric set multi-cover via quasi-uniform sampling. *JoCG*, 7(1):221–236, 2016. 11

[7] Reuven Bar-Yehuda, Magnús M. Halldórsson, Joseph Naor, Hadas Shachnai, and Irina Shapira. Scheduling split intervals. *SIAM J. Comput.*, 36(1):1–15, 2006. 2, 4

[8] Reuven Bar-Yehuda and Dror Rawitz. Using fractional primal-dual to schedule split intervals with demands. *Discrete Optimization*, 3(4):275–287, 2006. 2, 3

[9] Ayelet Butman, Danny Hermelin, Moshe Lewenstein, and Dror Rawitz. Optimization problems in multiple-interval graphs. *ACM Trans. Algorithms*, 6(2):40:1–40:18, 2010. 1, 2, 3, 5, 6, 8

[10] Timothy M. Chan, Elyot Grant, Jochen Könemann, and Malcolm Sharpe. Weighted capacitated, priority, and geometric set cover via improved quasi-uniform sampling. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2012, Kyoto, Japan, January 17-19, 2012*, pages 1576–1585, 2012. 3, 4, 6, 7, 8

[11] Timothy M. Chan and Sariel Har-Peled. Approximation Algorithms for Maximum Independent Set of Pseudo-Disks. *Discrete & Computational Geometry*, 48(2):373–392, 2012. 3, 4, 10, 11

[12] Chandra Chekuri, Kenneth L. Clarkson, and Sariel Har-Peled. On the set multicover problem in geometric settings. *ACM Trans. Algorithms*, 9(1):9:1–9:17, 2012. 4

[13] Chandra Chekuri, Kent Quanrud, and Zhao Zhang. On approximating partial set cover and generalizations. *CoRR*, abs/1907.04413, 2019. 11

[14] Kenneth L. Clarkson and Kasturi R. Varadarajan. Improved approximation algorithms for geometric set cover. *Discrete & Computational Geometry*, 37(1):43–58, 2007. 4

[15] Irit Dinur, Venkatesan Guruswami, Subhash Khot, and Oded Regev. A new multilayered PCP and the hardness of hypergraph vertex cover. *SIAM J. Comput.*, 34(5):1129–1146, 2005. 9

[16] Alina Ene, Sariel Har-Peled, and Benjamin Raichel. Geometric packing under nonuniform constraints. *SIAM J. Comput.*, 46(6):1745–1784, 2017. 11

[17] Matt Gibson and Imran A. Pirwani. Algorithms for dominating set in disk graphs: Breaking the $\log n$ barrier - (extended abstract). In *Algorithms - ESA 2010, 18th Annual European Symposium, Liverpool, UK, September 6-8, 2010. Proceedings, Part I*, pages 243–254, 2010. 4

[18] Jerrold R. Griggs and Douglas B. West. Extremal values of the interval number of a graph. *SIAM J. Matrix Analysis Applications*, 1(1):1–7, 1980. 2

[19] Dorit S. Hochbaum and Asaf Levin. Cyclical scheduling and multi-shift scheduling: Complexity and approximation algorithms. *Discrete Optimization*, 3(4):327–340, 2006. 3, 8

[20] Tanmay Inamdar and Kasturi R. Varadarajan. On partial covering for geometric set systems. In *34th International Symposium on Computational Geometry, SoCG 2018, June 11-14, 2018, Budapest, Hungary*, pages 47:1–47:14, 2018. 11

[21] Frank Kammer and Torsten Tholey. Approximation algorithms for intersection graphs. *Algorithmica*, 68(2):312–336, 2014. 2, 3

[22] Frank Kammer, Torsten Tholey, and Heiko Voepel. Approximation Algorithms for Intersection Graphs. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, 13th International Workshop, APPROX 2010, and 14th International Workshop, RANDOM 2010, Barcelona, Spain, September 1-3, 2010. Proceedings*, pages 260–273, 2010. 1, 2, 3, 4

[23] Subhash Khot. On the power of unique 2-prover 1-round games. In John H. Reif, editor, *Proceedings on 34th Annual ACM Symposium on Theory of Computing, May 19-21, 2002, Montréal, Québec, Canada*, pages 767–775. ACM, 2002. 3

[24] Subhash Khot and Oded Regev. Vertex cover might be hard to approximate to within 2-epsilon. *J. Comput. Syst. Sci.*, 74(3):335–349, 2008. 3, 9

[25] Nabil H. Mustafa and Kasturi R. Varadarajan. Epsilon-approximations and epsilon-nets. *CoRR*, abs/1702.03676, 2017. 4

[26] Aniket Basu Roy, Sathish Govindarajan, Rajiv Raman, and Saurabh Ray. Packing and covering with non-piercing regions. *Discrete & Computational Geometry*, 60(2):471–492, 2018. 4

[27] Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*, volume 24. Springer Science & Business Media, 2003. 9

[28] William P Thurston. *The geometry and topology of three-manifolds*. Princeton University Princeton, NJ, 1979. 2

[29] Kasturi R. Varadarajan. Weighted geometric set cover via quasi-uniform sampling. In *Proceedings of the 42nd ACM Symposium on Theory of Computing, STOC 2010, Cambridge, Massachusetts, USA, 5-8 June 2010*, pages 641–648, 2010. 3, 4, 7

[30] Yuli Ye and Allan Borodin. Elimination graphs. *ACM Trans. Algorithms*, 8(2):14:1–14:23, 2012. 2

AUTHORS

Chandra Chekuri
Professor, Department of Computer Science
University of Illinois, Urbana-Champaign
Urbana, Illinois, United States
chekuri@illinois.edu
https://chekuri.cs.illinois.edu/

Tanmay Inamdar[4]
Researcher, Department of Informatics,
University of Bergen
Bergen, Norway
Tanmay.Inamdar@uib.no

ABOUT THE AUTHORS

CHANDRA CHEKURI received his B. Tech in Computer Science and Engineering from Indian Institute of Technology, Madras (now Chennai) in 1993 and received his Ph. D. in Computer Science from Stanford University in 1998 under the supervision of the late Rajeev Motwani. After finishing his Ph. D. he worked eight formative years at Lucent Bell Labs before moving to the University of Illinois. He is interested in many topics but mainly works in algorithm design, combinatorial optimization, graphs, and mathematical programming.

TANMAY INAMDAR received his Ph. D. in Computer Science from The University of Iowa in 2020, where he was advised by Kasturi Varadarajan. Currently he is working as a Researcher at The University of Bergen in Norway. His research interests include Approximation Algorithms, Computational Geometry, and Parameterized Algorithms.

[4]This work was done when the author was a PhD student at The University of Iowa, during a visit to University of Illinois, Urbana-Champaign.