A note on the hardness of approximating the k-way Hypergraph Cut problem

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Abstract

We consider the approximability of k-way Hypergraph Cut problem: the input is an edge-weighted hypergraph $G = (V, \mathcal{E})$ and an integer k and the goal is to remove a min-weight subset of the edges such that the residual graph has at least k connected components. When G is a graph this problem admits a $2(1 - 1/k)$-approximation [8], however, there has been no non-trivial approximation ratio for general hypergraphs. In this note we show, via a very simple reduction, that an α-approximation for k-way Hypergraph Cut implies an $O(\alpha^2)$-approximation for the Densest k-Subgraph problem. This gives conditional hardness of approximation for k-way Hypergraph Cut since the best known approximation ratio for Densest k-Subgraph is $O(n^{1/4+\varepsilon})$ [1] and resolving its approximability is a major open problem. As a corollary we obtain conditional hardness for k-way Submodular Multiway Partition problem which generalizes k-way Hypergraph Cut. These resuls are in contrast to $2(1-1/k)$-approximation for closely related problems where the goal is to separate k given terminals [3, 4].

1 Introduction

We consider the following problem.

k-way Hypergraph Cut: Let $G = (V, \mathcal{E})$ be hypergraph with edge weights given by $w : \mathcal{E} \to \mathbb{R}_+$. Given an integer k, find a min-weight subset of edges $\mathcal{E}' \subseteq \mathcal{E}$ such that $G - \mathcal{E}'$ has at least k connected components. Equivalently find a partition of V into k non-empty sets $V_1, V_2, \ldots, V_k$ such that the weight of the hyperedges that cross the partition[1] is minimized.

k-way Hypergraph Cut is known as the k-Cut problem when the input is a graph and is one of the well-studied variants of graph partitioning problems. k-way Hypergraph Cut is a special case of a more general submodular partitioning problem defined below.

k-Way Submodular Partition (k-way Sub-MP): Let $f : 2^V \to \mathbb{R}_+$ be a non-negative submodular set function[2] over a finite ground set V. The k-way submodular partition problem is to find a partition $V_1, \ldots, V_k$ of V to minimize $\sum_{i=1}^{k} f(V_i)$ such that for $1 \le i \le k$, $V_i \neq \emptyset$. An important special case is when f is symmetric and we refer to it as k-way Sym-Sub-MP.

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[1] A hyperedge e crosses a partition of the vertex set if e properly intersects at least two parts of the partition.

[2] A set function $f : 2^V \to \mathbb{R}$ is submodular iff $f(A) + f(B) \ge f(A \cap B) + f(A \cup B)$ for all $A, B \subseteq V$. Moreover, f is symmetric if $f(A) = f(V - A)$ for all $A \subseteq V$.

We refer the reader to [3] to see why k-way Hypergraph Cut is a special case of k-way Sub-MP. The k-Cut problem is not only a special case of k-way Hypergraph Cut but it is also a special case of k-way Sym-Sub-MP. When k is part of the input all the problems we discussed so far are NP-Hard, and also APX-Hard to approximate. k-way Sym-Sub-MP admits a $2(1 - 1/k)$-approximation [7, 11] and hence also k-Cut [8]. For k-way Hypergraph Cut a $2\Delta(1 - 1/k)$-approximation easily follows from the $2(1 - 1/k)$-approximation for k-Cut; here Δ is the rank of the hypergraph (the maximum size of any hyperedge). On the other hand, in the general case, the known approximation algorithms for k-way Hypergraph Cut and k-way Sub-MP provide an approximation ratio of $\Omega(k)$ [11]. It was an open problem to obtain an improved understanding of their approximability.

In this note we show that a good approximation for k-way Hypergraph Cut would imply a good approximation for the Densest k-Subgraph problem. Resolving the approximability of Densest k-Subgraph is a well-known open problem. We first describe Densest k-Subgraph formally.

Densest k-Subgraph: Given a graph $G = (V, E)$ and integer k, find a subset $S \subseteq V$ of k nodes to maximize the edges in the induced graph $G[S]$.

The current best approximation for Densest k-Subgraph is $O(n^{1/4+\varepsilon})$ [1]; note that an $O(k)$-approximation is trivial. On the other hand we can rule out a PTAS for Densest k-Subgraph only under the the assumption that NP $\not\subseteq \cap_{\epsilon>0}$BPTIME$(2^{n^\epsilon})$ [6]. Moroever [2] shows polynomial-factor integrality gaps for several strong SDP relaxations. Resolving the wide gap in the approximability of Densest k-Subgraph is a major open problem. There have been several problems that have been shown to be conditionally hard by giving a reduction from Densest k-Subgraph which has further cemented the importance of Densest k-Subgraph as a canonical problem.

To formally state the implication of our reduction it is more convenient to relate the approximation ratio to the parameter s which is the sum of the number of nodes and edges. The tight instances for the algorithm of [1] for Densest k-Subgraph have $|E| = \Theta(|V|^{3/2})$. For these instances, it is not known how to obtain an approximation ratio better than $O(|V|^{1/4}) = O(s^{1/6})$.

Theorem 1.1 *A polynomial-time $\alpha(s)$ approximation algorithm for* k-way Hypergraph Cut *implies a polynomial-time $O(\alpha^2(s + 1))$-approximation algorithm for* Densest k-Subgraph.

When k is a fixed constant one can reduce k-way Hypergraph Cut and k-way Sub-MP to solving $O(n^{k-1})$ instances of the "terminal" version of these problems which have a $2(1 - 1/k)$ approximation. We refer the reader to [3, 4] for more details on these related problems.

2 Proof of Theorem 1.1

Let $(G = (V, E), \ell)$ be an instance of Densest k-Subgraph, where to avoid confusion, we use ℓ to denote the number of nodes in the subgraph we wish to find. We construct a hypergraph $H = (A, \mathcal{F})$ as follows. For each edge $e \in E$ we create a node a_e and add it to U. Moreover we add a new special node r to U. Thus $A = \{r\} \cup \{a_e \mid e \in E\}$. For each node $v \in V$ we add a hyperedge f_v to $\mathcal{F}$ where $f_v = \{r\} \cup \{a_e \mid e \in \delta_G(v)\}$ where $\delta_G(v)$ is the set of edges in E that are incident to v in G. Thus H is basically the hypergraph obtained from G by flipping the role of nodes and edges and then adding the extra node r to each hyperedge. We also observe that $|U| + |\mathcal{F}| = 1 + |V| + |E| = s + 1$. The basic and simple claim about G and H is the following.

Claim 2.1 *For any $1 \leq \ell \leq |V|$, if there is set $S \subseteq V$ with $|S| = \ell$ and $|E_G(S)| = L - 1$ then the* k-way Hypergraph Cut *instance on H with $k = L$ has a cut of value at most ℓ. Moreover, given any $F \subseteq \mathcal{F}$ with $|F| = \ell'$ such that $H - F$ has L' connected components then there is a subset $S' \subseteq V$ such that $|S'| = |F|$ and $|E_G(S')| = L' - 1$.*

Proof: Consider a set $F \subseteq \mathcal{F}$ of hyperedges in H. Suppose we remove them from H. Let $V_F = \{v \in V \mid f_v \in F\}$ be the nodes in G that correspond to the edges in F. Then a node $a_e \in A$ corresponding to an edge $e = uv$ is separated from r in H iff both $u, v \in V_F$; in this case the node a_e becomes an isolated node in $H - F$. Thus the number of connected components in $H - F$ is precisely equal to $|E_G(V_F)| + 1$. This correspondence proves both parts of the claim. □

Suppose we have an α approximation for k-way Hypergraph Cut. We will obtain an α^2-approximation for Densest k-Subgraph as follows. Let (G, ℓ) be a given instance of Densest k-Subgraph. First assume that we know the optimum solution value L for the given instance. We construct H the hypergraph as described and give H and $k = L + 1$ to the α-approximation algorithm for k-way Hypergraph Cut. By Claim 2.1 there is an optimum solution to the k-way Hypergraph Cut instance on H of value ℓ. Thus, the approximation algorithm will output a set $F \subseteq \mathcal{F}$ such that (i) $|F| \leq \alpha \cdot \ell$ and (ii) $H - F$ has at least $L + 1$ connected components. By the second part of the claim we can obtain a set $S' \subseteq V$ such that $|S'| \leq \alpha \cdot \ell$ and $|E_G(S')| \geq L$. A random subset S of S' where $|S| = \ell$ induces, in expectation, at least L/α^2 edges. One can derandomize this step. Thus we can obtain a set $S \subseteq V$ such that $|S| = \ell$ and $|E_G(S)| \geq L/\alpha^2$. Since L is the optimum value for the given instance of Densest k-Subgraph, we obtain the desired α^2-approximation. We can remove the assumption of the knowledge of L by trying all possible values of L from 0 to $|E(G)|$. This finishes the proof of Theorem 1.1.

3 Open problems

The main open question is to obtain a hardness of approximation for k-way Hypergraph Cut under the $P \neq NP$ assumption. At this point we only have APX-Hardness coming from k-Cut; we should note that APX-Hardness for k-Cut has been claimed by Papadimitriou (see [8]) but as far as we know no published proof has appeared in the literature. k-way Sub-MP appears to be much more general than k-way Hypergraph Cut so it may be easier to first establish hardness for k-way Sub-MP. In fact it may be feasible to prove strong unconditional lower bounds for k-way Sub-MP in the oracle model via the techniques from [9, 4].

When k is a fixed constant k-Cut can be solved in polynomial time [5]. k-way Hypergraph Cut is known to be solvable in polynomial time for $k \leq 3$ [10] but its status is open for any fixed $k \geq 4$. In fact we also do not know whether k-way Sym-Sub-MP or k-way Sub-MP are NP-Hard for any fixed $k > 2$.

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