SIAM J. DISCRETE MATH.
Vol. 28, No. 2, pp. 577–597

A GRAPH REDUCTION STEP PRESERVING ELEMENT-CONNECTIVITY AND PACKING STEINER TREES AND FORESTS*

CHANDRA CHEKURI† AND NITISH KORULA‡

Abstract. Given an undirected graph $G = (V, E)$ and a subset of vertices called terminals $T \subseteq V$, the *element-connectivity* $\kappa'_G(u, v)$ of two terminals $u, v \in T$ is the maximum number of u-v paths that are pairwise element-disjoint, that is, disjoint in both edges and nonterminals $V \setminus T$. (Element-connectivity was first (implicitly) defined by Frank, Ibaraki, and Nagamochi in [*J. Graph Theory*, 17 (1993), pp. 275–281].) (Element-disjoint paths need not be disjoint in terminals.) Hind and Oellermann [*Congr. Numer.*, 113 (1996), pp. 179–204] gave a graph reduction step that preserves the *global* element-connectivity of the terminals. We show that one can also apply such a reduction step while preserving *local* connectivity, that is, all the pairwise element-connectivities of the terminals. We illustrate the usefulness of this more general reduction step by giving applications to packing element-disjoint Steiner trees and forests: Given a graph G and disjoint terminal sets $T_1, T_2, \ldots, T_h$, we seek a maximum number of element-disjoint Steiner forests where each forest connects each T_i. We prove that if each T_i is k-element-connected, then there exist $\Omega(\frac{k}{\log |T| \log h})$ element-disjoint Steiner forests, where $T = \bigcup_i T_i$. If G is planar (or has fixed genus), we show that there exist $\Omega(k)$ Steiner forests. Our proofs are constructive, giving poly-time algorithms to find these forests; these are the first nontrivial algorithms for packing element-disjoint Steiner forests.

Key words. element-connectivity, Steiner trees, Steiner forests, element-disjoint, packing Steiner trees and forests

AMS subject classifications. 05C70, 05C40, 68R10, 05C85, 90C27

DOI. 10.1137/120882147

1. Introduction.

Menger [29] proved the following fundamental min-max relation on vertex-connectivity: Given an undirected graph $G(V, E)$ and two nonadjacent nodes u, v, the maximum number of internally vertex-disjoint paths in G between u and v is equal to the minimum number of vertices whose deletion separates u from v. Similarly, the maximum number of edge-disjoint paths connecting u and v is equal to the minimum number of edges whose deletion separates u from v. These are special cases of the well-known max-flow/min-cut theorem.

Hind and Oellermann [18] considered a natural generalization of Menger's theorem to more than two vertices: Given a graph $G(V, E)$ and a set of vertices $T \subseteq V$ called terminals, what is the maximum number of disjoint *trees* that connect T? As all such trees contain T, the question is meaningful if one asks for trees that are disjoint in *elements*, which consist of both the edges E and vertices of $V \setminus T$ (the nonterminals). In an equivalent formulation of the question, one may assume that T forms a stable set in G by subdividing any edge between terminals and ask for

*Received by the editors June 22, 2012; accepted for publication (in revised form) August 15, 2013; published electronically April 8, 2014. An extended abstract of this work originally appeared in [4]. The original publication is available at www.springerlink.com, and the copyright is held by Springer-Verlag.

http://www.siam.org/journals/sidma/28-2/88214.html

†Department of Computer Science, University of Illinois, Urbana, IL 61801 (chekuri@cs.illinois.edu). This author's research was partially supported by NSF grants CCF-0728782 and CNS-0721899.

‡Google Research, New York, NY 10011 (nitish@google.com). This author's research was done while the author was at the Department of Computer Science of the University of Illinois, and was partly supported by NSF grant CCF-0728782 and a University of Illinois Dissertation Completion Fellowship.

a set of trees that contain T but are disjoint in $V \setminus T$. Thus, the question of Hind and Oellermann can be stated as follows: What is the maximum number of *element-disjoint* trees connecting a set of terminals T? Here, the natural upper bound is the minimum number of elements whose deletion separates T; analogous to the definitions of vertex-connectivity and edge-connectivity, this number is referred to as the *element-connectivity* of T.

One might wish to check if a min-max relation analogous to Menger's theorem holds, i.e., if the number of element-disjoint trees connecting T is equal to the element-connectivity of T. Similarly, one might conjecture that the maximum number of *edge-disjoint* trees spanning T is equal to the edge-connectivity of T. However, neither of these conjectures is true even when $|T| = 3$: Consider K_4, the complete graph on four vertices, and let T be any set of three vertices. It is easy to see that the edge-connectivity and element-connectivity of T are 3, but there are only two edge-disjoint or element-disjoint trees connecting T. Hind and Oellermann [18] considered the case when $|T| \leq 4$ and showed that if T is k-element-connected, then there exist $\lfloor \frac{1}{|T|-1} \lceil \frac{|T|k}{2} \rceil \rfloor$ element-disjoint trees connecting T; moreover, this bound is tight and generalizes Menger's theorem when $|T| = 2$. They established their result via a useful graph *reduction step*. Subsequently, Cheriyan and Salavatipour [7] independently studied this question, calling this the problem of packing element-disjoint Steiner trees;[1] crucially using the same graph reduction step (which they rediscovered), they showed that if k is the element-connectivity of T, there always exist $\Omega(k/\log |T|)$ element-disjoint Steiner trees. Moreover, this bound is tight (up to constant factors) in the worst case when $|T|$ is large.

Another motivation for studying element-connectivity comes from network design problems. Jain et al. [22] first studied element-connectivity in this domain as a way to generalize some results in edge-connectivity network design. This approach has proved fruitful; results on element-connectivity network design have been crucially used in obtaining breakthroughs on vertex-connectivity network design [12, 8, 4, 30, 9].

The motivating applications and the discussion above show that element-connectivity is a useful bridge between edge-connectivity and vertex-connectivity. In this paper we make two contributions. We consider the graph reduction step for element-connectivity introduced by Hind and Oellermann [18] and prove that it preserves all local element-connectivities and not just the global element-connectivity. We then demonstrate the applicability of this new Reduction Lemma (with additional ideas) to pack element-disjoint Steiner forests, a generalization of the problem of packing element-disjoint Steiner trees. We discuss each of these in turn and formally state our results.

1.1. A graph reduction step preserving element connectivity. The well-known *splitting-off* operation introduced by Lovász [27] is a standard tool in the study of (primarily) edge-connectivity problems. Given an undirected multigraph G and two edges su and sv incident to s, the splitting-off operation replaces su and sv by the single edge uv. Lovász proved the following theorem on splitting off to preserve *global* edge-connectivity.

THEOREM 1.1 (Lovász). *Let $G = (V \cup \{s\}, E)$ be an undirected multigraph in which V is k-edge-connected for some $k \geq 2$ and the degree of s is even. Then for every edge su there is another edge sv such that V is k-edge-connected after splitting off su and sv.*

[1] A Steiner tree is simply a tree connecting all vertices of T.

Mader strengthened the above theorem to show the existence of two edges incident to s that, when split off, preserve the *local* edge-connectivity (that is, pairwise connectivity of all pairs of vertices) of the graph.

THEOREM 1.2 (Mader [28]). *Let $G = (V \cup \{s\}, E)$ be an undirected multigraph, where $deg(s) \neq 3$ and s is not incident to a cut edge of G. Then s has two neighbors u and v such that the graph G' obtained from G by replacing su and sv by uv satisfies $\lambda_{G'}(x, y) = \lambda_G(x, y)$ for all $x, y \in V \setminus \{s\}$.*

Generalizations to directed graphs are also known [28, 13, 20]. The splitting-off theorems have numerous applications in graph theory and combinatorial optimization; see [27, 14, 24, 21, 5, 26, 25, 23] for various pointers. Although splitting-off techniques can sometimes be used in the study of vertex-connectivity, their use is limited, and no generally applicable theorem akin to Theorem 1.2 is known. On the other hand, Hind and Oellermann [18] proved an elegant theorem on preserving global element-connectivity.

Given a graph $G(V, E)$, with its vertex set partitioned into a set T of terminals and a set $V \setminus T$ of nonterminals, we define the element-connectivity between two terminals u, v, denoted by $\kappa'_G(u, v)$, as the minimum number of elements (i.e., edges and nonterminals) whose deletion separates u from v. (Note that one could equivalently define $\kappa'_G(u, v)$ as the maximum number of element-disjoint paths between u and v.) We use $\kappa'_G(T) = \min_{u,v\in T} \kappa'_G(u, v)$ to denote the minimum number of elements whose deletion separates some terminals from others. We use G/pq to denote the graph obtained from G by contracting the edge pq.

THEOREM 1.3 (Hind and Oellermann [18]). *Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ be a terminal-set such that $\kappa'_G(T) \geq k$. Let (p, q) be* any *edge where $p, q \in V \setminus T$. Then $\kappa'_{G_1}(T) \geq k$ or $\kappa'_{G_2}(T) \geq k$, where $G_1 = G - pq$ and $G_2 = G/pq$.*

We generalize this theorem to show that either deleting an edge or contracting its endpoints preserves the *local* element-connectivity of every pair of terminals.

REDUCTION LEMMA. *Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ be a terminal set. Let (p, q) be* any *edge where $p, q \in V \setminus T$, and let $G_1 = G - pq$ and $G_2 = G/pq$. Then one of the following holds:*

(i) $\forall u, v \in T$, $\kappa'_{G_1}(u, v) = \kappa'_G(u, v)$.

(ii) $\forall u, v \in T$, $\kappa'_{G_2}(u, v) = \kappa'_G(u, v)$.

Remark 1. The Reduction Lemma, applied repeatedly, transforms a graph into another graph in which the nonterminals form a stable set. Moreover, the reduced graph is a minor of the original graph.

Theorem 1.3 has been useful in the study of element-connectivity and found applications in [7, 23]. The stronger Reduction Lemma, which preserves *local* connectivity, increases its applicability; we demonstrate applications (using additional ideas) to problems on packing Steiner trees and forests that we have briefly alluded to already; we discuss these below.

1.2. Packing element-disjoint Steiner trees and forests. There has been much interest in the recent past in algorithms for (integer) packing of disjoint Steiner trees in both the edge- and element-connectivity settings [24, 21, 26, 25, 6, 7, 5]. A *Steiner tree* is simply a tree containing the entire terminal set T. We do not explicitly mention the terminal set unless needed. See [17] for applications of Steiner tree packing to VLSI design. An important and motivating conjecture is the following.

CONJECTURE 1 (Kriesell). *Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ be a set of terminals that are $2k$-edge-connected in G. Then there are k edge-disjoint Steiner trees in G.*

Note that if $T = V$, then the conjecture is a corollary of the well-known theorem of Tutte and Nash-Williams on the maximum number of edge-disjoint spanning trees in a graph. Lau [26] proved that if T is $24k$-edge-connected, then there exist k edge-disjoint Steiner trees; previously it was not known if ck-connectivity for T implied k edge-disjoint Steiner trees for any fixed constant c. West and Wu improved the connectivity requirement to $6.5k$ [32]. Recall that for packing element-disjoint Steiner trees, Cheriyan and Salavatipur [7] showed that if T is k-element-connected, then there exist $\Omega(k/\log|T|)$-element-disjoint Steiner trees, and this is tight (up to constant factors) in the worst case. The algorithm in [7] to obtain the trees is randomized; a deterministic algorithm is given in [10]. It is also known that the problem of packing element-disjoint Steiner trees is hard to approximate to within an $\Omega(\log n)$ factor [6].

We consider the problem of packing Steiner *forests*. Given several disjoint terminal sets $T_1, T_2, \ldots T_h \subseteq V(G)$, a *Steiner forest* is a forest such that each T_i $(1 \leq i \leq h)$ is contained in a single component of the forest. Lau extended his results in [25] to show that if each set T_i is $32k$-edge-connected in the graph G, then G contains k edge-disjoint Steiner forests. We remark that Mader's splitting-off theorem plays an important role in Lau's work. In this paper we consider the problem of packing element-disjoint Steiner forests that was posed by [7]. The input consists of a graph $G = (V, E)$ and disjoint terminal sets $T_1, T_2, \ldots, T_h$, such that $\kappa'_G(T_i) \geq k$ for $1 \leq i \leq h$. What is the maximum number of element-disjoint forests such that in each forest, T_i is connected for $1 \leq i \leq h$? Our local connectivity reduction step is primarily motivated by this question. For general graphs we prove the following theorem.

THEOREM 1.4. *Let $T_1, \ldots, T_h$ be disjoint terminal sets in a graph $G = (V, E)$, and let $\kappa'_G(T_i) \geq k$ for $1 \leq i \leq h$. Then there exist $\Omega(k/(\log|T|\log h))$ element-disjoint Steiner forests, where $T = \bigcup_i T_i$. Moreover, there is a polynomial-time algorithm to output the Steiner forests.*

We also study the packing problem in planar graphs and graphs of fixed genus and prove a stronger result.

THEOREM 1.5. *Let $T_1, \ldots, T_h$ be disjoint terminal sets in a planar graph $G = (V, E)$, and let $\kappa'_G(T_i) \geq k$ for $1 \leq i \leq h$. Then there is a polynomial-time algorithm to find $\lceil k/5 \rceil - 1$ element-disjoint Steiner forests. If G has genus $g \geq 1$, then there is a polynomial-time algorithm to find $\Omega(k/g)$ element-disjoint Steiner forests. Moreover, the degree of each nonterminal in each of the forests is 2.*

These are the first nontrivial bounds for packing element-disjoint Steiner forests in general graphs or planar graphs. Since element-connectivity generalizes edge-connectivity, our bounds in planar graphs are stronger than those given by Lau [26, 25] for *edge*-connectivity. Our proof is also simple; this simplicity comes from thinking about element-connectivity (using the Reduction Lemma) instead of edge-connectivity!

COROLLARY 1.6. *Let $T_1, \ldots, T_h$ be disjoint terminal sets in a planar graph $G = (V, E)$, and let $\lambda_G(T_i) \geq k$ for $1 \leq i \leq h$. Then there exists a polynomial-time algorithm to find $\lceil k/5 \rceil - 1$ edge-disjoint Steiner forests.*

Other related work. Aazami, Cheriyan, and Jampani [1] showed that in planar graphs, if T is k-element-connected, then there always exist $k/2 - 1$ element-disjoint Steiner trees. They complement their upper bound by showing that in planar graphs, for any fixed $\varepsilon > 0$, it is NP-hard to obtain a $(1/2 + \varepsilon)$-approximation to the problem of finding the maximum number of element-disjoint Steiner trees. They also generalized their result on planar graphs; if G excludes an H-minor, then there are $\Omega(k)$

element-disjoint Steiner trees where the constant depends on the size of H. These results are based on the use of Theorem 1.3 and a result of Frank, Király, and Kriesell [16] on packing spanning trees in hypergraphs. The approach in [1] via [16] leads to stronger results than ours for packing Steiner trees in planar and minor-free graphs; however, it does not (appear to) apply to packing Steiner forests even with our new Reduction Lemma for preserving local connectivity. Our work on planar graphs was done independently of [1], although we were inspired to study these graphs by a question of Joseph Cheriyan.

Organization. We prove the Reduction Lemma in section 2 and use it to get a polylogarithmic approximation for packing element-disjoint Steiner forests in section 3. Finally, in section 4, we obtain better approximations for packing Steiner trees and forests in graphs with simpler structure, such as planar, low-genus, or low-treewidth graphs.

2. The Reduction Lemma.

Let $G(V,E)$ be a graph, with a given set $T \subseteq V(G)$ of terminals. For ease of notation, we subsequently refer to terminals as *black* vertices and nonterminals (also called Steiner vertices) as *white*. The elements of G are white vertices and edges; two paths are *element-disjoint* if they have no white vertices or edges in common. Recall that the element-connectivity of two black vertices u and v, denoted by $\kappa'_G(u,v)$, is the maximum number of element-disjoint (that is, disjoint in edges and white vertices) paths between u and v in G. We omit the subscript G when it is clear from the context.

For this section, to simplify the proof, we will assume without loss of generality that G has no edges between black vertices; any such edge can be subdivided, with a white vertex inserted between the two black vertices. It is easy to see that two paths are element-disjoint in the original graph iff they are element-disjoint in the modified graph. Thus, we can say that paths are element-disjoint if they share no white vertices or that u and v are k-element-connected if the smallest set of white vertices whose deletion separates u from v has size k.

Recall that our lemma generalizes Theorem 1.3 on preserving global connectivity. We remark that our proof is based on a cutset argument unlike the path-based proofs in [18, 7] for the global case.

Reduction Lemma. *Given $G(V,E)$ and T, let $pq \in E(G)$ be any edge such that p and q are both white. Let $G_1 = G - pq$ and $G_2 = G/pq$ be the graphs formed from G by deleting and contracting pq, respectively. Then,*

(i) $\forall u,v \in T, \kappa'_{G_1}(u,v) = \kappa'_G(u,v)$, *or*

(ii) $\forall u,v \in T, \kappa'_{G_2}(u,v) = \kappa'_G(u,v)$.

Proof. Consider an arbitrary edge pq between two white nodes. Deleting or contracting an edge can reduce the element-connectivity of a pair by at most 1. Suppose the lemma were not true; there must be pairs s,t and x,y of black vertices such that $\kappa'_{G_1}(s,t) = \kappa'_G(s,t) - 1$ and $\kappa'_{G_2}(x,y) = \kappa'_G(x,y) - 1$. The pairs have to be distinct since it cannot be the case that $\kappa'_{G_1}(u,v) = \kappa'_{G_2}(u,v) = \kappa'_G(u,v) - 1$ for any pair u,v. (To see this, if one of the $\kappa'_G(u,v)$ u-v paths uses pq, contracting the edge will not affect that path and will leave the other paths untouched. Otherwise, no path uses pq, and so it can be deleted.) Note that one of s,t could be the same vertex as one of x,y; for simplicity, we consider this case later and focus first on the case that $\{s,t\} \cap \{x,y\} = \emptyset$. We show that our assumption on the existence of s,t and x,y with the above properties leads to a contradiction. Let $\kappa'_G(s,t) = k_1$ and $\kappa'_G(x,y) = k_2$. We use the following facts several times.

1. Any cutset of size less than k_1 that separates s and t in G_1 cannot include p or q. (If it did, it would also separate s and t in G.)

2. $\kappa'_{G_1}(x, y) = k_2$ since $\kappa'_{G_2}(x, y) = k_2 - 1$.

We define a vertex tripartition of a graph G as follows: (A, B, C) is a vertex tripartition of G if A, B, and C partition $V(G)$, B contains only white vertices, and there are no edges between A and C. (That is, removing the white vertices in B disconnects A and C.)

Since $\kappa'_{G_1}(s, t) = k_1 - 1$, there is a vertex tripartition (S, M, T) in G_1 such that $|M| = k_1 - 1$ and $s \in S$ and $t \in T$. From fact 1 above, M cannot contain p or q. For the same reason, it is also easy to see that p and q cannot both be in S (or both be in T); otherwise M would be a cutset of size $k_1 - 1$ in G. Therefore, assume without loss of generality that $p \in S, q \in T$.

Similarly, since $\kappa'_{G_2}(x, y) = k_2 - 1$, there is a vertex tripartition (X, N', Y) in G_2 with $|N'| = k_2 - 1$ and $x \in X$ and $y \in Y$. We claim that N' contains the contracted vertex pq, for otherwise N' would be a cutset of size $k_2 - 1$ in G. Therefore, it follows that (X, N, Y) where $N = N' \cup \{p, q\} - \{pq\}$ is a vertex tripartition in G that separates x from y. Note that $|N| = k_2$ and N includes *both* p and q. For the latter reason we note that (X, N, Y) is a vertex tripartition also in G_1.

Subsequently, we work with the two vertex tripartitions (S, M, T) and (X, N, Y) in G_1 (we stress that we work in G_1 and not in G or G_2). Recall that $s, p \in S$, and $t, q \in T$, and M has size $k_1 - 1$; also, N separates x from y and $p, q \in N$. Figure 2.1(a) shows these vertex tripartitions. Since M and N contain only white vertices, all terminals are in S or T and in X or Y. We say that $S \cap X$ is *diagonally opposite* from $T \cap Y$, and $S \cap Y$ is diagonally opposite from $T \cap X$. Let A, B, C, D denote $S \cap N, X \cap M, T \cap N$, and $Y \cap M$, respectively, with I denoting $N \cap M$; note that A, B, C, D, I partition $M \cup N$.

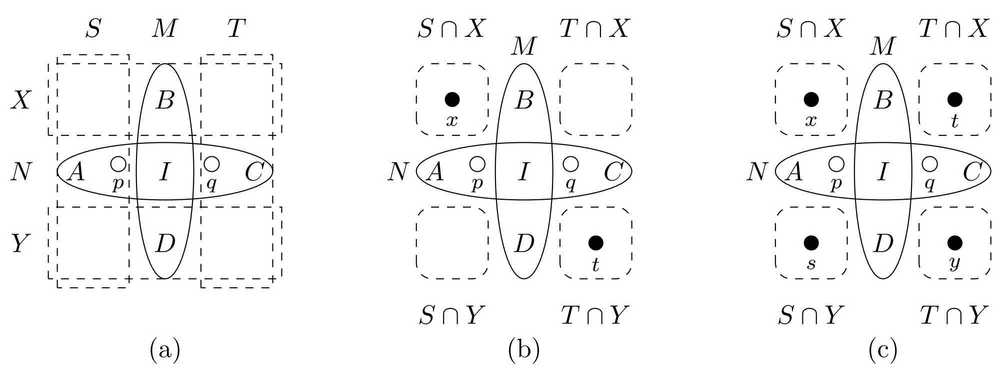

FIG. 2.1. *Part* (a) *illustrates the vertex tripartitions* (S, M, T) *and* (X, N, Y). *In parts* (b) *and* (c), *we consider possible locations of the terminals* s, t, x, y.

CLAIM 1. *Neither s nor t can be diagonally opposite from either x or y.*

Proof. Suppose, for contradiction, that x and t are diagonally opposite; the other cases are similar. Then, we have $x \in S \cap X$ and $t \in T \cap Y$. Figure 2.1(b) illustrates this case. Observe that $A \cup I \cup B$ separates x from y (regardless of whether $y \in S \cap Y$ or $y \in T \cap Y$); since x and y are k_2-connected in G_1 and $A \cup I \cup C$ (which is the set N) has size k_2, it follows that $|B| \geq |C|$. Similarly, $C \cup I \cup D$ separates t from s, and since C contains q, fact 1 implies that $|C \cup I \cup D| \geq k_1$. But $M = B \cup I \cup D$ and $|M| = k_1 - 1$. Therefore, $|C| > |B|$, yielding a contradiction. ∎

Claim 1 restricts possible locations of the four vertices. We assume without loss of generality that $x \in S$, and hence $x \in S \cap X$. Since t cannot be diagonally opposite from x, $t \in T \cap X$. Similarly, we cannot have y diagonally opposite from t, and so $y \in T \cap Y$. And s and y cannot be diagonally opposite, so we have $s \in S \cap Y$. Figure 2.1(c) shows the required positions of the vertices. Now, N separates s from t and contains p, q; therefore, from fact 1, $|N| \geq k_1 > |M|$. But M separates x from y, and fact 2 implies that x, y are k_2-connected in G_1; therefore, $|M| \geq k_2 = |N|$, and we have a contradiction.

Recall that we have focused on the case that s, t, x, y represent four distinct vertices. Suppose one of $\{s, t\}$ is the same vertex as one of $\{x, y\}$; without loss of generality, let $x = s$. This vertex is in $S \cap X$; by Claim 1, neither y (because of s) nor t (because of x) can be in $T \cap Y$. Therefore, $y \in S \cap Y$ and $t \in T \cap X$; but this makes them diagonally opposite, again giving a contradiction. ∎

3. Packing Steiner trees and forests in general graphs. Consider a graph $G(V, E)$, with its vertex set V partitioned into $T_1, T_2, \ldots T_h, W$. We refer to each T_i as a group of *terminals* and to W as the set of Steiner or white vertices; we use $T = \bigcup_i T_i$ to denote the set of all terminals. A Steiner forest for this graph and this partition is a forest that is a subgraph of G, such that each T_i is entirely contained in a single tree of this forest. (Note that T_i and T_j can be in the same tree.) For any group T_i of terminals, we define $\kappa'(T_i)$, the element-connectivity of T_i, as the largest k such that for every $u, v \in T_i$, the element-connectivity of u and v in the graph G is at least k.

We say two Steiner forests for G are element-disjoint if they share no edges or Steiner vertices. (Every Steiner forest must contain all the terminals.) The Steiner forest packing problem is to find as many element-disjoint Steiner forests for G as possible. By inserting a Steiner vertex between any pair of adjacent terminals, we can assume that there are no edges between terminals, and then the problem of finding element-disjoint Steiner forests is simply that of finding Steiner forests that do not share any Steiner vertices. A special case is when $h = 1$, in which case we seek a maximum number of element-disjoint Steiner trees.

PROPOSITION 3.1. *If $k = \min_i \kappa'_G(T_i)$, there are at most k element-disjoint Steiner forests in G.*

Proof. Let S be a set of k white vertices that separates vertices u and v in T_i. Any tree that contains both u and v must contain a vertex of S. Hence, we can pack at most k trees that contain all of T_i. ∎

Cheriyan and Salavatipour [7] proved that if there is a single group T of terminals, with $\kappa'(T) = k$, then there always exist $\Omega(k/\log|T|)$ Steiner trees. Their algorithm proceeds by using Theorem 1.3, the global element-connectivity reduction of [18], to delete and contract edges between Steiner vertices, while preserving $\kappa'(T) = k$. Then, once we obtain a bipartite graph G' with terminals on one side and Steiner vertices on the other side, we randomly color the Steiner vertices using $k/6\log|T|$ colors; they show that with high probability, each color class connects the terminal set T, giving $k/6\log|T|$ trees. The bipartite case can be cast as a special case of packing bases of a polymatroid, and a variant of the random coloring idea is applicable in this more general setting [10]; a derandomization is also provided in [10], thus yielding a deterministic polynomial-time algorithm to find $\Omega(k/\log|T|)$ element-disjoint Steiner trees.

In this section, we give algorithms for packing element-disjoint Steiner forests, where we are given h groups of terminals $T_1, T_2, \ldots, T_h$. The approach of [7] encounters two difficulties. First, we cannot reduce to a bipartite instance, using only the global-connectivity version of the Reduction Lemma. In fact, our strengthening of the Reduction Lemma to preserve local connectivity was motivated by this; using it allows us once again to assume that we have a bipartite graph $G'(T \cup W, E)$. Second, we cannot apply the random coloring algorithm on the bipartite graph G' directly; we give an example in section A.1 to show that this approach does not work. One reason for this is that, unlike the Steiner tree case, it is no longer a problem of packing bases of a submodular function. To overcome this second difficulty we use a decomposition technique followed by the random coloring algorithm to prove that there always exist $\Omega(k/(\log |T| \log h))$ element-disjoint forests. We believe that the bound can be improved to $\Omega(k/\log |T|)$.

In order to pack element-disjoint Steiner forests, we borrow the basic idea from [5] in the *edge-connectivity* setting for Eulerian graphs; this idea was later used by Lau [25] in the much more difficult non-Eulerian case. The idea at a high level is as follows: If all the terminals are k-connected, then we can treat the terminals as forming one group and reduce the problem to that of packing Steiner trees. Otherwise, we can find a cut $(S, V \setminus S)$ that separates some groups from others. If the cut is chosen appropriately, we may be able to treat one side, say, S, as containing a single group of terminals and pack Steiner *trees* in them *without* using the edges crossing the cut. Then we can shrink S and find Steiner forests in the reduced graph; unshrinking S is possible since we have many trees on S. In [5, 25] this scheme works to give $\Omega(k)$ edge-disjoint Steiner forests. However, the approach relies strongly on properties of edge-connectivity as well as the properties of the packing algorithm for Steiner trees. These do not generalize easily for element-connectivity. Nevertheless, we show that the basic idea can be applied in a slightly weaker way (resulting in the loss of an $O(\log h)$ factor over the Steiner tree packing factor). We remark that the reduction to a bipartite instance using the Reduction Lemma plays a critical role. A key definition is the notion of a good separator given below.

Definition 3.2. *Given an graph $G(V, E)$ with terminal sets $T_1, T_2, \ldots T_h$, such that for all i, $\kappa'(T_i) \geq k$, we say that a set S of white vertices is a* good separator *if* (i) $|S| \leq k/2$ *and* (ii) *there is a component of $G - S$ in which all terminals are $k/(2 \log h)$-element-connected.*

Note that the empty set is a good separator if all terminals are $k/(2 \log h)$-element-connected.

Lemma 3.3. *For any instance of the Steiner forest Packing problem, there is a polynomial-time algorithm that finds a good separator.*

Proof. Let $G(V, E)$ be an instance of the Steiner forest packing problem, with terminal sets $T_1, T_2, \ldots T_h$ such that each T_i is k-element-connected. If T is $\frac{k}{2 \log h}$-element connected (which can be easily checked in polynomial time), the empty set S is a good separator.

Otherwise, there is some set of white vertices of size less than $\frac{k}{2 \log h}$ that separates some of the terminals from others. Let S_1 be a minimal such set, and consider the two or more components of $G - S_1$. Note that each T_i is entirely contained in a single component, since T_i is at least k-element-connected, and $|S_1| < k$. Among the components of $G - S_1$ that contain terminals, consider a component G_1 with the fewest sets of terminals; G_1 must have at most $h/2$ sets from $T_1, \ldots T_h$. If the set of all terminals in G_1 is $\frac{k}{2 \log h}$ connected, we stop; otherwise, we find in G_1 a set of white vertices S_2 with size less than $\frac{k}{2 \log h}$ that separates terminals of G_1. Again, we find a component G_2 of $G_1 - S_2$ with fewest sets of terminals and repeat this procedure until we obtain some subgraph G_ℓ in which all the terminals are $\frac{k}{2\log h}$-connected. We can always find such a subgraph, since the number of sets of terminals is decreasing by a factor of 2 or more at each stage, so we find at most $\log h$ separating sets S_j. Now, we observe that the set $S = \bigcup_{j=1}^{\ell} S_j$ is a good separator. It separates the terminals in G_ℓ from the rest of T, and its size is at most $\log h \times \frac{k}{2\log h} = k/2$; it follows that each set of terminals T_i is entirely within G_ℓ or entirely outside it. By construction, all terminals in G_ℓ are $\frac{k}{2\log h}$-connected. To see that this algorithm runs in polynomial time, it suffices to observe that if the sets of white vertices $S_1, S_2, \ldots$ exist, they can be found in polynomial time via min-cut algorithms. Once no such set S_ℓ exists in $G_{\ell-1}$, we have found the desired separator. □

We can now prove our main result of this section, which is that we can always find a packing of $\Omega(\frac{k}{\log|T|\log h})$ Steiner forests.

THEOREM 3.4. *Given a graph $G(V,E)$, with terminal sets $T_1, T_2, \ldots T_h$, such that for all i, $\kappa'(T_i) \geq k$, there is a polynomial-time algorithm to pack $\Omega(k/\log|T|\log h)$ element-disjoint Steiner forests in G.*

Proof. The proof is by induction on h. The base case of $h = 1$ follows from [7, 10]; G contains at least $\frac{k}{6\log|T|}$ element-disjoint Steiner *trees*, and we are done.

We may assume G is bipartite by using the Reduction Lemma. Find a good separator S, and a component G_ℓ of $G - S$ in which all terminals are $\frac{k}{2\log h}$-connected. Now, since the terminals in G_ℓ are $\frac{k}{2\log h}$-connected, use the algorithm of [7] to find $\frac{k}{12\log h\log|T|}$ element-disjoint Steiner trees containing all the terminals in G_ℓ; none of these trees uses vertices of S. Number these trees from 1 to $\frac{k}{12\log h\log|T|}$; let $\mathcal{T}_j$ denote the jth tree.

The set S separates G_ℓ from the terminals in $G - G_\ell$. If S is not a minimal such set, discard vertices until it is. If we delete G_ℓ from G and add a clique between the white vertices in S to form a new graph G', it is clear that the element-connectivity between any pair of terminals in G' is at least the element-connectivity they had in G. The graph G' has $h' \leq h - 1$ groups of terminals; by induction, we can find $\frac{k}{12\log|T|\log h} < \frac{k}{12\log|T|\log h'}$ element-disjoint Steiner forests for the terminals in G'. As before, number the forests from 1 to $\frac{k}{12\log h\log|T|}$; we use $\mathcal{F}_j$ to refer to the jth forest. These Steiner forests may use the newly added edges between the vertices of S; these edges do not exist in G. However, we claim that the Steiner forest $\mathcal{F}_j$ of G', together with the Steiner tree $\mathcal{T}_j$ in G_ℓ, gives a Steiner forest of G. The only way this might not be true is if $\mathcal{F}_j$ uses some edge added between vertices $u, v \in S$. However, by the minimality of S, every vertex in S is adjacent to a terminal in G_ℓ, and all the terminals of G_ℓ are in every one of the Steiner trees we generated. Therefore, there is a path from u to v in $\mathcal{T}_j$. Hence, deleting the edge between u and v from $\mathcal{F}_j$ still leaves each component of $\mathcal{F}_j \cup \mathcal{T}_j$ connected.

Therefore, for each $1 \leq j \leq \frac{k}{12\log h\log|T|}$, the vertices in $\mathcal{F}_j \cup \mathcal{T}_j$ induce a Steiner forest for G. To see that this algorithm runs in polynomial time, note that we can find a good separator S in polynomial time, and we then recurse on the graph G'. As G' has $h' \leq h - 1$ groups of terminals, there are at most $h \leq n$ levels of recursion. □

4. Packing Steiner trees and forests in planar graphs. We now prove much improved results for restricted classes of graphs, in particular planar graphs. If G is planar, we show the existence of $\lceil k/5 \rceil - 1$ element-disjoint Steiner forests.[2]

[2]Note that in the special case of packing Steiner trees, the paper of Aazami, Cheriyan, and Jampani [1] shows that there are $\lfloor k/2 \rfloor - 1$ element-disjoint Steiner trees.

The (simple) technique extends to graphs of fixed genus to prove the existence of $\Omega(k)$ Steiner forests where the constant depends mildly on the genus. We believe that there exist $\Omega(k)$ Steiner forests in any H-minor-free graph where H is fixed; it is shown in [1] that there exist $\Omega(k)$ Steiner *trees* in H-minor-free graphs. Our technique for planar graphs does not extend directly, but generalizing this technique allows us to make partial progress; by using our general graph result and some related ideas, in section 4.2, we prove that in graphs of any fixed treewidth, there exist $\Omega(k)$ element-disjoint Steiner trees if the terminal set is k-element-connected.

The intuition and algorithm for planar graphs are easier to describe for the Steiner tree packing problem, and we do this first. We achieve the improved bound by observing that planarity restricts the use of many white vertices as "branch points" (that is, vertices of degree ≥ 3) in forests. Intuitively, even in the case of packing trees, if there are terminals $t_1, t_2, t_3, \ldots$ that must be in every tree, and white vertices $w_1, w_2, w_3 \ldots$ that all have degree 3, it is difficult to avoid a $K_{3,3}$-minor.[3] Note, however, that degree 2 white vertices behave like edges and do not form an obstruction. We capture this intuition more precisely by showing that there must be a pair of terminals t_1, t_2 that are connected by $\Omega(k)$ degree 2 white vertices; we can contract these "parallel edges" and recurse.

We describe below an algorithm for packing Steiner trees. Throughout the rest of this section, we assume $k > 10$; otherwise, $\lceil k/5 \rceil - 1 \leq 1$, and we can always find one Steiner tree in a connected graph.

Given an instance of the Steiner tree packing problem in planar graphs, we construct a *reduced instance* as follows: Use the Reduction Lemma to delete and contract edges between white vertices to obtain a planar graph with vertex set $T \cup W$, such that W is a stable set. Now, for each vertex $w \in W$ of degree 2, connect the two terminals that are its endpoints directly with an edge, and delete w. (All edges have unit capacity.) We now have a planar *multigraph*, though the only parallel edges are between terminals, as these were the only edges added while deleting degree 2 vertices in W. Note that this reduction preserves the element-connectivity of each pair of terminals; further, any set of element-disjoint trees in this reduced instance corresponds to a set of element-disjoint trees in the original instance.

LEMMA 4.1. *In a reduced instance of the planar Steiner tree packing problem, if each vertex in T has degree at least k, there are two terminals t_1, t_2 with at least $\lceil k/5 \rceil - 1$ parallel edges between them.*

We defer a complete proof of this lemma, which is somewhat intricate, to section A.2, but at the end of this subsection, we present a much simpler argument to show that there exist terminals t_1, t_2 with $\lceil k/10 \rceil$ edges between them. First, though, we show that Lemma 4.1 allows us to pack $\lceil k/5 \rceil - 1$ disjoint trees.

THEOREM 4.2. *Given an instance of the Steiner tree packing problem on a planar graph G with terminal set T, if $\kappa'(T) \geq k$, there is a polynomial-time algorithm to find at least $\lceil k/5 \rceil - 1$ element-disjoint Steiner trees in G. Moreover, in each tree, the white (nonterminal) vertices all have degree* 2.

Proof. We prove this theorem by induction on $|T|$; if $|T| = 2$, there are k disjoint *paths* in G from one terminal to the other, so we are done (including the guarantee of degree 2 for white vertices).

Otherwise, apply the Reduction Lemma to construct a reduced instance G', preserving the element-connectivity of T. Now, from Lemma 4.1, there exists a pair of terminals t_1, t_2 that have $\lceil k/5 \rceil - 1$ parallel edges between them (note that the parallel

[3]Strictly speaking, this is not true, though the intuition is helpful.

edges between t_1 and t_2 may have nonterminals on them in the original graph, but they have degree 2). Contract t_1, t_2 into a single terminal t, and consider the new instance of the Steiner tree packing problem with terminal set $T' = T \cup \{t\} - \{t_1, t_2\}$. It is easy to see that the element-connectivity of the terminal set is still at least k; by induction, we can find $\lceil k/5 \rceil - 1$ Steiner trees containing all the terminals of T', with the property that all nonterminals have degree 2. Taking these trees together with $\lceil k/5 \rceil - 1$ edges between t_1 and t_2 gives $\lceil k/5 \rceil - 1$ trees in G' that span the original terminal set T. □

It now remains only to prove Lemma 4.1, which we do in section A.2; here, we show the weaker, but considerably easier, result that there are two terminals with $\lceil k/10 \rceil$ parallel edges between them. We need the following technical result.

THEOREM 4.3 (Borodin [2]). *If G is a planar graph with minimum degree 3, it has an edge of weight at most 13, where the weight of an edge is the sum of the degrees of its endpoints.*

We do not provide a proof of this well-known theorem; it is easy to verify using a straightforward discharging argument. (The proof of Lemma A.1 in section A.2 is similar, and that is what we use to obtain the stronger result guaranteeing $\lceil k/5 \rceil - 1$ parallel edges.)

Proof of Lemma 4.1 (weaker version). Let G be the planar multigraph of the reduced instance; every terminal has degree at least k in G. Construct a planar graph G' from G by keeping only a single copy of each edge. We argue below that some terminal $t_1 \in T$ has degree at most 10 in G'; it follows that G must contain at least $\lceil k/10 \rceil$ copies of some edge incident to t_1, as t_1 has degree at least k in G. These edges must be incident to another terminal t_2, completing the proof.

To see that some terminal t_1 has degree at most 10 in G', we first assume that no terminal has degree ≤ 2, or we are already done. Now, as every nonterminal in a reduced instance has degree at least 3, we may use Theorem 4.3; this implies that G' has an edge e, such that the sum of the degrees of the endpoints of e is at most 13. The edge e must be incident to a terminal t_1, as the nonterminals are a stable set. The other endpoint of e has degree at least 3, so the degree of t_1 is at most 10. □

4.1. Packing Steiner forests in planar graphs.

For the planar Steiner forest packing problem, we use an algorithm very similar to that for packing Steiner trees above. Now, as input, we are given sets $T_1, \dots T_m$ of terminals that are each internally k-connected, but some T_i and T_j may be poorly connected. The algorithm described above for packing Steiner trees encounters a technical difficulty when we try to extend it to Steiner forests. Lemma 4.1 can be used at the start to merge some two terminals. Precisely as before, as long as each T_i contains at least two terminals, it is internally k-element-connected, and hence all terminals have degree at least k. Therefore, Lemma 4.1 is true, and we can contract some pair of terminals t_1, t_2 that have $\lceil k/5 \rceil - 1$ parallel edges between them. Note that if t_1, t_2 are in the same T_i, after contraction, we have an instance in which T_i contains fewer terminals, and we can apply induction. If t_1, t_2 are in different sets T_i, T_j, then, after contracting, all terminals in T_i and T_j are pairwise k-connected, so we can merge these two groups into a single set.

However, as the algorithm proceeds it may get stuck in the following situation: it merges all terminals from some group T_i into a single terminal. Now this terminal does not require any more connectivity to other terminals, although other groups are not yet merged together. In this case we term this terminal as dead. In proving the crucial Lemma 4.1, we argued that in the multigraph G of the reduced instance, every terminal has degree at least k (since it is k-element-connected to other terminals), and in the graph G' in which we keep only a single copy of each edge, some terminal has degree at most 10; therefore, there are $\lceil k/10 \rceil$ copies of some edge. However, in the Steiner forest problem, some T_i may contain only a single dead terminal t (after several contraction steps). The terminal t may be poorly connected to the remaining terminals; therefore, it may have degree less than k in the multigraph G. If t is the unique low-degree terminal in G', we may not be able to find a pair of terminals with a large number of edges between them. Thus, in the presence of dead terminals, Lemma 4.1 no longer applies; we illustrate this with a concrete example at the end of section A.1.

We solve this problem by eliminating a set T_i when it has only a single dead terminal t. One cannot simply delete this terminal or replace it by a single white vertex, as several paths connecting other terminals may pass through t. Instead, we replace the dead terminal t with a "well-linked" collection of white vertices so that distinct paths through t can now use disjoint white vertices from this collection. It might be most natural to replace t by a clique of white vertices, but this would not preserve planarity; instead, we replace a dead terminal t with a grid of white vertices, which ensures that the resulting graph is still planar. We then apply the Reduction Lemma to remove edges between the newly added white vertices and proceed with the merging process. We formalize this intuition in the following lemma.

LEMMA 4.4. *Let $G(V, E)$ with a given $T \subseteq V$ be a planar graph and $t \in T$ be an arbitrary terminal of degree d. Let G' be the graph constructed from G by deleting t and inserting a $d \times d$ grid of white vertices, with the edges incident to t in G made incident to distinct vertices on one side of the new grid in G'. Then the following hold:*

1. *G' is planar.*
2. *For every pair u, v of terminals in G', $\kappa'_{G'}(u, v) = \kappa'_G(u, v)$.*
3. *Any set of element-disjoint subgraphs of G' corresponds to a set of element-disjoint subgraphs of G.*

Proof. See Figure 4.1, showing this operation; it is easy to observe that given a planar embedding of G, one can construct a planar embedding of G'. It is also clear that a set of element-disjoint subgraphs in G' corresponds to such a set in G; every subgraph that uses a vertex of the grid can contain the terminal t.

It remains only to argue that the element-connectivity of every other pair of terminals is preserved. Let u, v be an arbitrary pair of terminals; we show that their element-connectivity in G' is at least their connectivity $\kappa'(u, v)$ in G. Fix a set of $\kappa'(u, v)$ paths in G from u to v; let $\mathcal{P}$ be the paths that use the terminal t, and let $\ell = |\mathcal{P}|$. We locally modify these $\ell \leq d$ paths in $\mathcal{P}$ by routing them through the grid, so we obtain $\kappa'(u, v)$ element-disjoint paths in G'.

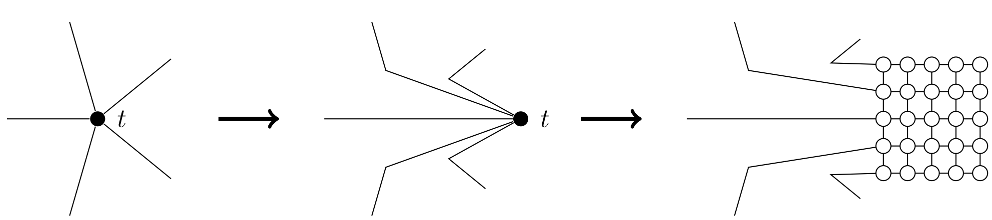

FIG. 4.1. *Replacing a terminal by a grid of white vertices preserves planarity and element-connectivity.*

Let $\mathcal{P}_u$ denote the set of prefixes from u to t of the ℓ paths in $\mathcal{P}$, and let $\mathcal{P}_v$ denote the suffixes from t to v of these paths. Let H denote the $d \times d$ grid that replaces t in G'; we use $\mathcal{P}'_u$ and $\mathcal{P}'_v$ to denote the corresponding paths in G' from u to vertices of H, and from vertices in H to v, respectively. Let $\mathcal{I}$ and $\mathcal{O}$ denote the vertices of H incident to paths in $\mathcal{P}'_u$ and $\mathcal{P}'_v$. It is not difficult to see that there is a set of disjoint paths in the grid H connecting the ℓ distinct vertices in $\mathcal{I}$ to those in $\mathcal{O}$; using the paths of $\mathcal{P}'_u$, together with the paths through H and the paths of $\mathcal{P}'_v$, gives us a set of disjoint paths in G' from u to v. □

Extensions. Our result for planar graphs can be generalized to graphs of fixed genus; Ivančo [19] generalized Theorem 4.3 to show that a graph G of genus g has an edge of weight at most $2g + 13$ if $0 \leq g \leq 3$ and an edge of weight at most $4g + 7$ otherwise. This allows us to prove that there exist $\lceil k/c \rceil$ forests where $c \leq 4g + 8$; we have not attempted to optimize this constant c. Aazami, Cheriyan, and Jampani [1] also give algorithms for packing Steiner trees in these graph classes and graphs excluding a fixed minor. We thus make the following natural conjecture.

CONJECTURE 2. *Let $G = (V, E)$ be an H-minor-free graph, with terminal sets $T_1, T_2, \ldots T_m$, such that for all i, $\kappa'(T_i) \geq k$. There exist $\Omega(k/c)$ element-disjoint Steiner forests in G, where c depends only on the size of H.*

We note that Lemma 4.1 fails to hold for H-minor-free graphs and in fact fails even for bounded treewidth graphs. Thus, our approach cannot be directly generalized. However, instead of attempting to contract together just two terminals connected by many parallel edges, we may be able to contract together a constant number of terminals that are "internally" highly connected. Using Theorem 3.4 and other ideas, we prove in the next section that this approach suffices to pack many trees in graphs with small treewidth. We believe that these ideas together with the structural characterization of H-minor-free graphs by Robertson and Seymour [31] should lead to a positive resolution of Conjecture 2.

4.2. Packing trees in graphs of bounded treewidth. Let $G(V, E)$ be a graph of treewidth $\leq r - 1$, with terminal set $T \subseteq V$ such that $\kappa'(T) \geq k$. In this section, we give an algorithm to find, for any fixed r, $\Omega(k)$ element-disjoint Steiner trees in G. Our approach is similar to that for packing Steiner trees in planar graphs, where we argued in Lemma 4.1 that there exist two terminals t_1, t_2 with $\Omega(k)$ parallel edges between them, so we could contract them together and recurse on a smaller instance. In graphs of bounded treewidth, this is no longer the case; see the end of the appendix for an example in which no pair of terminals is connected by many parallel edges. However, we argue that there exists a small set of terminals $T' \subset T$ that is highly "internally connected," so we can find $\Omega(k)$ disjoint trees connecting all terminals in T' without affecting the connectivity of terminals in $T - T'$. We can then contract together T' and the white vertices used in these trees to form a single new terminal t and again recurse on a smaller instance. The following lemma captures this intuition.

LEMMA 4.5. *If $G(V, E)$ is a bipartite graph of treewidth at most $r - 1$, with terminal set $T \subset V$ such that $T \geq 2^r$, $\kappa'(T) \geq k$, there exists a set $S \subseteq V - T$ such that there is a component G' of $G - S$ containing $k/12r^2 \log(3r)$ element-disjoint Steiner trees for the (at least two) terminals in G'. Moreover, these trees in G' can be found in polynomial time.*

Given this lemma, we prove below that for any fixed r, we can pack $\Omega(k)$ element-disjoint trees in graphs of treewidth at most $r - 1$. The proof combines ideas of Theorems 4.2 and 3.4.

THEOREM 4.6. *Let $G(V, E)$ be a graph of treewidth at most $r-1$. For any terminal set $T \subseteq V$ with $\kappa'_G(T) \geq k$, there exist $\Omega(k/12r^2 \log(3r))$ element-disjoint Steiner trees on T.*

Proof. As for Theorem 4.2, we prove this theorem by induction. Let G be a graph of treewidth at most $r-1$, with terminal set T. If $|T| \leq 2^r$, we have $k/6 \log |T| \geq k/6r$ element-disjoint trees from the tree-packing algorithm of Cheriyan and Salavatipour [7] in *arbitrary* graphs.

Otherwise, we use the Reduction Lemma to ensure that G is bipartite. Let S be a set of white vertices guaranteed to exist from Lemma 4.5. If S is not a minimal such set, discard vertices until it is. Now, find $k/12r^2 \log(3r)$ element-disjoint trees containing all terminals in some component G' of $G - S$; note that each vertex of S is incident to some terminal in G' and hence to every tree. (This follows from the minimality of S and the fact that G is bipartite.) Modify G by contracting all of G' to a single terminal t, and make it incident to every vertex of S. It is easy to see that all terminals in the new graph are k-element-connected; therefore, we now have an instance of the Steiner tree packing problem on a graph with fewer terminals. The new graph has treewidth at most $r-1$, so by induction, we have $k/12r^2 \log(3r)$ element-disjoint trees for the terminals in this new graph; taking these trees together with the $k/12r^2 \log(3r)$ trees of G' gives $k/12r^2 \log(3r)$ trees of the original graph G. ∎

We devote the rest of this section to proving the crucial Lemma 4.5. Subsequently, we may assume without loss of generality (after using the Reduction Lemma) that the graph G is bipartite; we may further assume that $k \geq 12r^2 \log(3r)$ and $|T| \geq 2^r$. First, observe that G has a small cutset that separates a few terminals from the rest.

PROPOSITION 4.7. *G has a cutset C of size at most r such that the union of some components of $G - C$ contains between r and $2r$ terminals.*

Proof. Fix a (rooted) tree-decomposition $\mathcal{T}$ of G. Every nonleaf node of $\mathcal{T}$ corresponds to a cutset, and each node of $\mathcal{T}$ contains at most r vertices of G. Let v be a deepest node in $\mathcal{T}$ such that the subtree rooted at each child of v has no more than $2r$ terminals. The nodes of G contained in v clearly form a cutset C of size at most r. If any subtree of $\mathcal{T}$ rooted at a child of v contains at least r terminals not contained in C, we are done. Otherwise, greedily select children of v until the total number of terminals in the associated subtrees not contained in C is between r and $2r$. ∎

We find the set S and component of $G - S$ in which we contract together a small number of terminals by focusing on the cutset C and components of $G - C$ that are guaranteed to exist from the previous proposition. We introduce some notation before proceeding with the proof.

1. Let C be a cutset of size at most r, and let V' be the vertices of the union of some components of $G - C$ containing between r and $2r$ terminals in total.
2. Since terminals in V' are k-connected to the terminals in the rest of the graph, and $|C| \leq r \ll k$, C contains at least one black vertex. Let C' be the set of black vertices in C.
3. Let $G' = G[V' \cup C']$ be the graph induced by V' and C'.

We omit a proof of the following straightforward proposition; the second part of the statement follows from the fact that each terminal in V' is k-connected to terminals outside G', and these paths to terminals outside G' must go through the cutset C of size at most r.

PROPOSITION 4.8. *The graph G' contains between r and $3r$ terminals (as C' may contain up to r terminals), and each terminal in V' is at least k/r-connected to some terminal in C'.*

Let T' be the set of terminals in G'. If $\kappa'_{G'}(T') \geq k/2r^2$, we can easily find a set of white vertices satisfying Lemma 4.5: Let S be the set of vertices of G that are adjacent (in G) to vertices of G'. It is obvious that S separates G' from the rest of G, and all terminals in T' are highly connected; from the tree packing result of [7], we can find the desired disjoint trees in G'. Finally, note that all vertices of S are white, as the only neighbors of G' are either white vertices of the cutset C or the neighbors of the black vertices in C, all of which are white as G is bipartite.

However, it may not be the case that all terminals of T' are highly connected in G'. In this event, we use the following simple algorithm (very similar to that in the proof of Lemma 3.3) to find a highly connected subset of T': Begin by finding a set S_1 of at most $k/2r^2$ white vertices in $G_0 = G'$ that separates terminals of T'. Among the components of $G_0 - S_1$, pick a component G_1 with at least one terminal of V'. If all terminals of G_1 are $k/2r^2$ connected, stop; otherwise, find in G_1 a set S_2 of at most $k/2r^2$ white vertices that separates terminals of G_1, pick a component G_2 of $G_1 - S_2$ that contains at least one terminal of V', and proceed in this manner until finding a component G_ℓ in which all terminals are $k/2r^2$-connected.

CLAIM 2. *We perform at most r iterations of this procedure before we stop, having found some subgraph G_ℓ in which all the (at least two) terminals are $k/2r^2$ connected.*

Proof. At least one terminal of C' must be lost every time we find such a set S_i; if this is true, the claim follows. To see that this is true, consider each iteration $i \geq 0$ before iteration r. Observe that when we find a cutset S_{i+1} in G_i, there is a component that we do *not* pick that contains a terminal t. If this terminal t is in C', we are done; otherwise, it must be in V'. But from Proposition 4.8 all terminals in V' were initially k/r-connected to some terminal in C', and we have deleted only $i+1 \leq r$ sets of white vertices, each of which contained at most $k/2r^2$ vertices. Therefore, some terminal of C' must be in the same component as t. When we stop with the subgraph G_ℓ, it contains at least one terminal $t' \in V'$ and at least one terminal of C' to which t' is highly connected; therefore, G_ℓ contains at least two terminals. ∎

All terminals in the subgraph G_ℓ are $k/2r^2$-connected, and there are at most $3r$ of them, so we can find $k/12r^2 \log(3r)$ disjoint trees *in* G_ℓ that connect them, using the tree packing result of [7]. Let S be the set of vertices of G that are adjacent (in G) to vertices of G_ℓ; obviously, S separates G_ℓ from the rest of G, and to satisfy Lemma 4.5, it merely remains to verify that S contains only white vertices. Every terminal in $G' - G_\ell$ was separated from G_ℓ by white vertices in some S_i, and terminals in $G - G'$ can only be incident to white vertices of the cutset C, which are not in G', let alone G_ℓ. This completes the proof of Lemma 4.5.

5. Conclusions.

We showed that the reduction step of [18] applies to local element-connectivity and demonstrated applications of this stronger Reduction Lemma to packing element- (and edge-) disjoint Steiner trees and forests. We close with several open questions:

(i) We believe that our bound on the number of element-disjoint Steiner forests in a general graph can be improved from $\Omega(k/(\log |T| \log h))$ to $\Omega(k/\log |T|)$.

(ii) It should be possible to prove Conjecture 2, on packing disjoint Steiner forests in graphs excluding a fixed minor. Chekuri and Ene [3] extended the techniques of this paper to show that one can pack $\Omega(k)$ element-disjoint Steiner forests in graphs of fixed treewidth, providing further evidence for the conjecture.

(iii) In a natural generalization of the Steiner forest packing problem, each non-terminal or white vertex has a *capacity*, and the goal is to pack forests subject to these capacity constraints. In general graphs it is easy to reduce this problem to the uncapacitated/unit-capacity version (for example, by replacing a white vertex of capacity c by a clique of size c), but this is not necessarily the case for restricted classes of graphs. In particular, it would be interesting to pack $\Omega(k)$ forests for the capacitated planar Steiner forest problem. An obvious first step is to prove this for packing element-disjoint Steiner trees in planar graphs. It is likely that this is possible, as one can *fractionally* pack $\Omega(k)$ element-disjoint Steiner trees in capacitated planar graphs; this follows from the work of Demaine, Hajiaghayi, and Klein [11].

Appendix. Packing element-disjoint trees and forests.

A.1. A counterexample to the random coloring algorithm for packing Steiner forests. We first define a graph H_k, which we use subsequently. H_k has two black vertices, x and y, and k white vertices, each incident to both x and y. (That is, there are k disjoint paths of white vertices from x to y.) Given a graph G, we define the operation of inserting H_k along an edge $pq \in E(G)$ as follows: Add the vertices and edges of H_k to G, delete the edge pq, and add edges from p to x and q to y. (If we collapsed H_k to a single vertex, we would have subdivided the edge pq.) Figure A.1 shows H_4 and the effect of inserting H_4 along an edge.

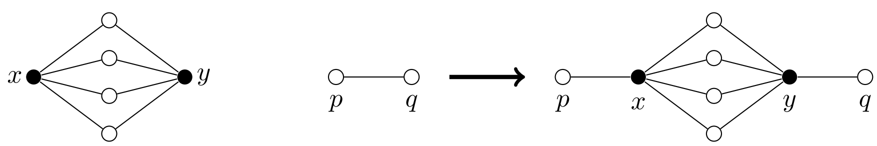

FIG. A.1. *On the left, the graph H_4. On the right, inserting it along a single edge pq.*

We now describe the construction of our counterexample. We begin with two black vertices, s and t, and k vertex-disjoint paths between them, each of length $k+1$; there are no edges other than those just described. Each of the k^2 vertices other than s and t is white. It is obvious that s and t are k-element-connected in this graph. Now, to form our final graph G_k, insert a copy of H_k along each of the $k(k-1)$ edges between a pair of white vertices. Figure A.2 shows the construction of G_3.

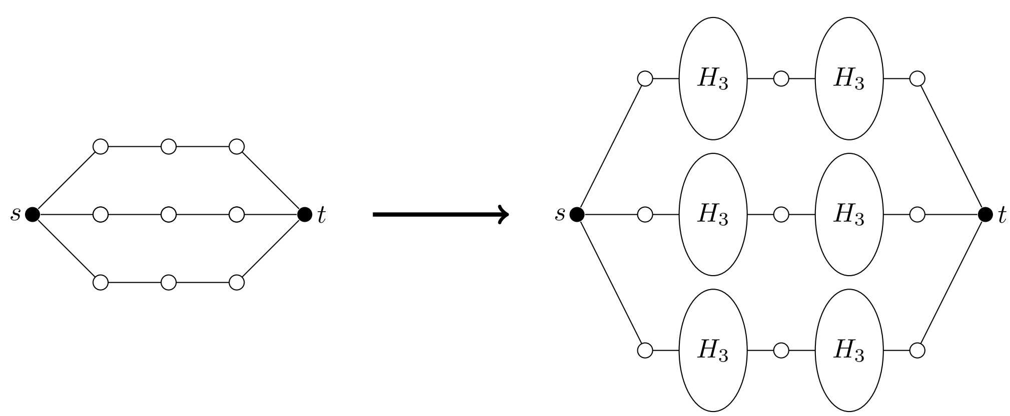

FIG. A.2. *The construction of G_3.*

The following claims are immediate:
(i) The vertices s and t are k-element-connected in G_k.
(ii) For every copy of H_k, the vertices x and y are k-white connected in G_k.

(iii) The graph G_k is bipartite, with the white vertices and the black vertices forming the two parts.

We use G_k as an instance of the Steiner forest packing problem; s and t form one group of terminals, and for each copy of H_k, the vertices x and y of that copy form a group. From our claims above, each group is k-element-connected.

If we use the algorithm of Cheriyan and Salavatipour, there are no edges between white vertices to be deleted or contracted, so we move directly to the coloring phase. If colors are assigned to the white vertices randomly, it is easy to see that no color class is likely to connect s and t. The probability that a white vertex is given color i is $\frac{c \log |T|}{k}$ for some constant c. The vertices s and t can be connected iff the same color is assigned to all the white vertices on one of the k paths from s to t in the graph formed from G_k by contracting each H_k to a single vertex. The probability that *every* vertex on such a path will receive the same color is $(\frac{c \log |T|}{k})^k$; using the union bound over the k paths gives us the desired result.

A counterexample to Lemma 4.1 *for planar Steiner forest.* Recall that in section 4.1, we pointed out that in the presence of dead terminals (after all terminals in some T_i have been contracted to a single vertex), Lemma 4.1 may no longer apply. As a concrete example, consider the graph G_k defined at the beginning of this appendix. (See also Figure A.1, and note that G_k is planar.) We have one terminal set $T_1 = \{s, t\}$ and other sets T_i containing the two terminals of each copy of H_k. After several contraction steps, each copy of H_k may have been contracted together to form a single terminal; each such terminal is only 2-connected to the rest of the graph. In the reduced instance, there is only a single copy of each edge, and Lemma 4.1 does not hold.

A.2. A tighter bound for planar graphs.

LEMMA A.1. *Let $G(T \cup W, E)$ be a planar graph with minimum degree* 3*, in which W is a stable set. There exists a vertex $t \in T$ of degree at most* 10*, with at most five neighbors in T.*

Proof. Our proof uses the *discharging* technique. Assume, for the sake of contradiction, that every vertex $t \in T$ has degree at least 11 or has at least 6 neighbors in T. By multiplying Euler's formula by 4, we observe that for a planar graph $G(V, E)$ with face set F, $(2|E| - 4|V|) + (2|E| - 4|F|) = -8$. We rewrite this as $\sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (l(f) - 4) = -8$, where $d(v)$ and $l(f)$ denote the degree of vertex v and length of face f, respectively.

Now, in our given graph G, assign $d(v) - 4$ units of *charge* to each vertex $v \in T \cup W$, and assign $l(f) - 4$ units of charge to each face f: Note that the net charge on the graph is negative (it is equal to -8). We describe rules for redistributing the charge through the graph such that after redistribution, if every terminal $t \in T$ has degree at least 11 or has at least 6 neighbors in T, the charge at each vertex and face will be nonnegative. But no charge is added or removed (it is merely rearranged), and so we obtain a contradiction.

We use the following rules for distributing charge:

1. Every terminal $t \in T$ distributes $1/3$ unit of charge to each of its neighbors in W.

2. Every terminal $t \in T$ distributes $1/2$ unit of charge to each triangular face f it is incident to, unless the face contains 3 terminals. In this case, it distributes $1/3$ unit of charge to the face.

We now observe that every vertex of W and every face has nonnegative charge. Each vertex $u \in W$ has degree at least 3 (the graph has minimum degree 3), so its initial charge was at least -1. It did not give up any charge, and rule 1 implies that it received $1/3$ from each of its (at least 3) neighbors, all of which are in T. Therefore, u has nonnegative charge after redistribution. If a face f has length 4 or more, it already had nonnegative charge, and it did not give up any. If f is a triangle, it starts with charge -1. It is incident to at least 2 terminals, since W is a stable set; we argue that it gains one unit of charge, to end with charge 0. From rule 2, if f is incident to two terminals, it gains $1/2$ unit from each of them, and if it is adjacent to 3 terminals, it gains $1/3$ unit from each of them.

It remains only to argue that each terminal $t \in T$ has nonnegative charge after redistribution. Consider a terminal of degree d; let b denote its number of terminal neighbors, Δ the number of triangular faces it is incident to, and Δ_3 the number of triangular faces containing 3 terminals. The charge that t has after redistribution is equal to

$$d - 4 - \frac{d-b}{3} - \frac{\Delta}{2} + \frac{\Delta_3}{6},$$

where $d-b$ is the number of neighbors in W, and we subtract $\Delta/2$ for every incident triangular face; the last term compensates for the fact that triangular faces with 3 terminals receive only $1/3$ units of charge from t.

To simplify these expressions, we will use the fact that since every neighbor in b can participate in at most 2 triangular faces with t, we have $b \geq \lceil \Delta/2 \rceil$. Also, we (trivially) have $\Delta \leq d$. To see that the charge left on t is always nonnegative, we consider a few cases.

First, suppose $d \geq 12$. The total charge left on t is equal to

$$\begin{aligned}\frac{2d}{3} - 4 + \frac{b}{3} - \frac{\Delta}{2} + \frac{\Delta_3}{6} &\geq \frac{2d}{3} - 4 + \frac{\Delta}{6} - \frac{\Delta}{2} \\ &= \frac{2d}{3} - 4 - \frac{\Delta}{3} \geq \frac{d}{3} - 4 \geq 0,\end{aligned}$$

where the first inequality follows from $b \geq \Delta/2$, the second from $\Delta \leq d$, and the final one from $d \geq 12$.

Now, suppose $d = 11$. If $\Delta \leq 10$, the same analysis gives a charge of at least $2d/3 - 4 - 10/3 = 0$. Therefore, we must consider the case when $\Delta = 11$. Now, we use the fact that $b \geq \lceil \Delta/2 \rceil = 6$, and since at least 6 out of t's 11 neighbors are terminals, there exists at least one triangle containing 3 terminals; that is, $\Delta_3 \geq 1$. Hence, the total charge left on t is at least

$$\frac{2d}{3} - 4 + \frac{b}{3} - \frac{\Delta}{2} + \frac{\Delta_3}{6} \geq \frac{22}{3} - 4 + \frac{6}{3} - \frac{11}{2} + \frac{1}{6} = 0.$$

Finally, we consider the case $d \leq 10$. Recall that from the hypothesis, t has at least 6 terminal neighbors, that is, $b \geq 6$. We use the following claim: $\Delta_3 \geq 2b - 2d + \Delta$. To see that this is true, consider the b terminal neighbors of t arranged around t. Moving around t, how could each of the b consecutive pairs not be part of a triangular face with t, and thus not contribute to Δ_3? There are only two possibilities: Either a consecutive pair of terminal neighbors is part of a nontriangular face with t, or the two vertices are not part of a face with t at all; in the latter case, there must be a neighbor $w \in W$ of t "in between" this pair of vertices. Figure A.3 illustrates these cases. In the former case, each of the $d - \Delta$ nontriangular faces incident to t can decrease Δ_3 by 1. In the latter case, each of the $d - b$ neighbors in W can decrease Δ_3 by 1. Therefore, $\Delta_3 \geq b - (d - \Delta) - (d - b) = 2b - 2d + \Delta$.

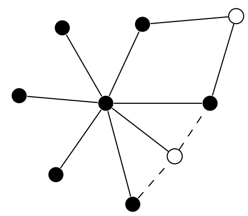

FIG. A.3. *Lower bounding the number of triangular faces with 3 terminals. Vertices of T are in black, and vertices of W are in white. Two "consecutive" terminals around t do not form a triangular face with t if they are part of a larger face, or if a vertex of W appears "between" them.*

Recall that we have $b \geq 6$ and that the charge left on t is equal to

$$\frac{2d}{3} - 4 + \frac{b}{3} - \frac{\Delta}{2} + \frac{\Delta_3}{6} \geq \frac{2d}{3} - 4 + \frac{b}{3} - \frac{\Delta}{2} + \frac{2b - 2d + \Delta}{6}$$
$$\geq \frac{2d}{3} - \frac{\Delta}{2} + \frac{\Delta - 2d}{6} = \frac{d}{3} - \frac{\Delta}{3} \geq 0,$$

where the first inequality follows from our claim above, the following inequality from $b \geq 6$, and the final inequality from $\Delta \leq d$. ∎

Proof of Lemma 4.1. Our argument is very similar to that of the proof in section 4 in that there are two terminals with at least $\lceil k/10 \rceil$ edges between them, except that here we use Lemma A.1 instead of Theorem 4.3.

Let G be the planar multigraph of the reduced instance; every terminal has degree at least k in G. Construct a planar graph G' from G by keeping a single copy of each edge; from Lemma A.1 above, some terminal t has degree at most 10 and at most 5 black neighbors. Let w denote the number of white neighbors of t and b the number of black neighbors. Since each white vertex is incident to only a single copy of each edge in G, there must be at least $\lceil (k-w)/b \rceil$ copies in G of some edge between t and a black neighbor. But $b \leq 5$ and $b + w \leq 10$; it is easy to verify that since $k \geq 10$, the smallest possible value of $\lceil (k-w)/b \rceil$ is $\lceil (k-5)/5 \rceil = \lceil k/5 \rceil - 1$. ∎

A.3. A counterexample to the existence of two terminals connected by $\Omega(k)$ "parallel edges". Recall that in the case of planar graphs (or graphs of bounded genus), we argued that there must be two terminals t_1, t_2 with $\Omega(k)$ "parallel edges" between them. (That is, there are $\Omega(k)$ degree-2 white vertices adjacent to t_1 and t_2.) This is not necessarily the case even in graphs of treewidth 3: The graph $K_{3,k}$, the complete bipartite graph with three vertices on one side and k on the other, has treewidth 3. If the three vertices on one side are the terminal set T and the k vertices of the other side are nonterminals, it is easy to see that $\kappa'(T) = k$, but every white vertex has degree 3.

In this example, there are only three terminals, so the tree packing algorithm of Cheriyan and Salavatipour [7] would allow us to find $\Omega(k/\log |T|) = \Omega(k)$ trees connecting them. Adding more terminals incident to all the white vertices would raise the treewidth, so this example does not immediately give us a low-treewidth graph with a large terminal set such that there are few parallel edges between any pair of terminals. However, we can easily extend the example by defining a graph G_m as follows: Let $T_1, T_2, \ldots, T_m$ be sets of two terminals each, let $W_1, W_2, \ldots, W_{m-1}$ each be sets of k white vertices, and let all the vertices in each W_i be adjacent to both terminals in T_i and both terminals in T_{i+1}. (See Figure A.4.) The graph G_m has $2m$ terminals, $T = \bigcup_i T_i$ is k-element-connected, and it is easy to verify that G_m has treewidth 4. However, every white vertex has degree 4, so there are no "parallel edges" between terminals. (One can modify this example to construct a counterexample graph G_m with treewidth 3 by removing one terminal from each alternate T_i.)

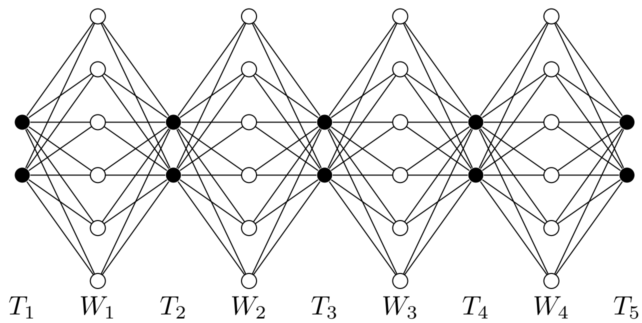

FIG. A.4. *A graph of treewidth* 4 *with many terminals but no "parallel edges."*

Acknowledgments. We thank Joseph Cheriyan for asking about planar packing of Steiner trees, which inspired our work on that problem. We also thank Oleg Borodin, Dan Cranston, Alexandr Kostochka, and Doug West for pointers on structural results on planar graphs. We thank the anonymous referees for their careful reading of the paper and numerous valuable suggestions, particularly on improving the proofs of section 4.1 and Lemmas 4.5 and A.1.

REFERENCES

[1] A. Aazami, J. Cheriyan, and K. Jampani, *Approximation algorithms and hardness results for packing element-disjoint Steiner trees in planar graphs*, Algorithmica, 63 (2012), pp. 425–456.

[2] O.V. Borodin, *Coupled colorings of graphs on a plane*, Metody Diskret. Analiz, 45 (1987), pp. 21–27 (in Russian).

[3] C. Chekuri and A. Ene, *Personal communication*, 2009.

[4] C. Chekuri and N. Korula, *A graph reduction step preserving element-connectivity and applications*, in Proceedings of the 36th International Colloquium on Automata, Languages and Programming, 2009, pp. 254–265.

[5] C. Chekuri and F.B. Shepherd, *Approximate integer decompositions for undirected network design problems*, SIAM J. Discrete Math., 23 (2008), pp. 163–177.

[6] J. Cheriyan and M.R. Salavatipour, *Hardness and approximation results for packing Steiner trees*, Algorithmica, 45 (2006), pp. 21–43.

[7] J. Cheriyan and M.R. Salavatipour, *Packing element-disjoint Steiner trees*, ACM Trans. Algorithms, 3 (2007), 47.

[8] J. Chuzhoy and S. Khanna, *Algorithms for single-source vertex connectivity*, in Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science, 2008, pp. 105–114.

[9] J. Chuzhoy and S. Khanna, *An* $O(k^3 \log n)$*-approximation algorithm for vertex-connectivity survivable network design*, Theory Comput., 8 (2012), pp. 401–413.

[10] G. Călinescu, C. Chekuri, and J. Vondrák, *Disjoint bases in a polymatroid*, Random Structures Algorithms, 35 (2009), pp. 418–430.

[11] E. Demaine, M.T. Hajiaghayi, and P. Klein, *Node-weighted Steiner tree and group Steiner tree in planar graphs*, in Proceedings of the 36th International Colloquium on Automata, Languages and Programming (ICALP), 2009, pp. 328–340.

[12] L. Fleischer, K. Jain, and D.P. Williamson, *Iterative rounding* 2*-approximation algorithms for minimum-cost vertex connectivity problems*, J. Comput. System Sci., 72 (2006), pp. 838–867.

[13] A. Frank, *On connectivity properties of Eulerian digraphs*, in Graph Theory in Memory of G. A. Dirac, Ann. Discrete Math. 41, North–Holland, Amsterdam, 1989, pp. 179–194.
[14] A. Frank, *Augmenting graphs to meet edge-connectivity requirements*, SIAM J. Discrete Math., 5 (1992), pp. 25–53.
[15] A. Frank, T. Ibaraki, and H. Nagamochi, *On sparse subgraphs preserving connectivity properties*, J. Graph Theory, 17 (1993), pp. 275–281.
[16] A. Frank, T. Király, and M. Kriesell, *On decomposing a hypergraph into k connected sub-hypergraphs*, Discrete Appl. Math., 131 (2003), pp. 373–383.
[17] M. Grötschel, A. Martin, and R. Weismantel, *The Steiner tree packing problem in VLSI design*, Math. Program., 78 (1997), pp. 265–281.
[18] H.R. Hind and O. Oellermann, *Menger-type results for three or more vertices*, Congr. Numer., 113 (1996), pp. 179–204.
[19] J. Ivančo, *The weight of a graph*, in Fourth Czechoslavakian Symposium on Combinatorics, Graphs and Complexity, Ann. Discrete Math. 51, North–Holland, Amsterdam, 1992, pp. 113–116.
[20] B. Jackson, *Some remarks on arc-connectivity, vertex splitting, and orientation in graphs and digraphs*, J. Graph Theory, 12 (2006), pp. 429–436.
[21] K. Jain, M. Mahdian, and M. Salavatipour, *Packing Steiner trees*, in Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), 2003, pp. 266–274.
[22] K. Jain, I. Măndoiu, V.V. Vazirani, and D.P. Williamson, *A primal-dual schema based approximation algorithm for the element connectivity problem*, in Proceedings of the 10th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), 1999, pp. 484–489.
[23] T. Király and L.C. Lau, *Approximate min-max theorems for Steiner rooted-orientations of graphs and hypergraphs*, J. Combin. Theory Ser. B, 98 (2008), pp. 1233–1252.
[24] M. Kriesell, *Edge-disjoint trees containing some given vertices in a graph*, J. Combin. Theory Ser. B, 88 (2003), pp. 53–65.
[25] L.C. Lau, *Packing Steiner forests*, in Proceedings of the 11th International Conference on Integer Programming and Combinatorial Optimization (IPCO), 2005, pp. 362–376.
[26] L.C. Lau, *An approximate max-Steiner-tree-packing min-Steiner-cut theorem*, Combinatorica, 27 (2007), pp. 71–90.
[27] L. Lovász, *On some connectivity properties of Eulerian graphs*, Acta Math. Hungarica, 28 (1976), pp. 129–138.
[28] W. Mader, *A reduction method for edge-connectivity in graphs*, in Advances in Graph Theory, Ann. Discrete Math. 3, North–Holland, Amsterdam, 1978, pp. 145–164.
[29] K. Menger, *Zur allgemeinen kurventheorie*, Fund. Math., 10 (1927), pp. 96–115.
[30] Z. Nutov, *An almost $O(\log k)$-approximation for k-connected subgraphs*, in Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), 2009, pp. 912–921.
[31] N. Robertson and P.D. Seymour, *Graph minors:* XVII. *Taming a vortex*, J. Combin. Theory Ser. B, 77 (1999), pp. 162–210.
[32] D. B. West and H. Wu, *Packing of Steiner trees and S-connectors in graphs*, J. Combin. Theory Ser. B, 102 (2012), pp. 186–205.