Contention Resolution for the ℓ-fold union of a matroid via the correlation gap

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Abstract

The correlation gap of a real-valued set function $f : 2^N \to \mathbb{R}_+$ [ADSY10] measures the worst-case ratio between two continuous extensions of f over all points in the unit cube; informally the gap measures the worst-case benefit of correlations between the variables.

This notion plays an important role in several areas including algorithms for constrained submodular function maximization via contention resolution schemes, mechanism design, and stochastic optimization. The correlation gap of any monotone submodular set function is known to be at least $(1 - 1/e)$, and this bound is tight even for the rank function of a uniform matroid of rank 1. Via a connection established in [CVZ14], this yields an optimal contention resolution scheme for rounding in a matroid polytope.

In this paper, we study the correlation gap of the rank function of the ℓ-fold union of a matroid $\mathcal{M}$, denoted by $\mathcal{M}^\ell$, defined as the (matroid) union of ℓ-copies of $\mathcal{M}$. We prove that the correlation gap of $\mathcal{M}^\ell$, for *any* matroid $\mathcal{M}$, is at most $1 - \frac{\ell^\ell e^{-\ell}}{\ell!}$; this bound behaves as $1 - \frac{1}{\sqrt{2\pi\ell}}$ as ℓ grows. This generalizes the results in [Yan11, BFGG22, KS23]. They established this gap for the uniform matroid of rank ℓ which can be viewed as the ℓ-fold union of a uniform matroid of rank 1; moreover this bound is tight even for this special case. The correlation gap yields a corresponding contention resolution scheme for $\mathcal{M}^\ell$ which was the initial motivation for this work.

1 Introduction

This paper is concerned with the correlation gap of the rank function of matroids, and its applications, in particular, to contention resolution schemes. We start with basics of matroids which can be skipped by a reader familiar with them. A matroid is a pair $\mathcal{M} = (N, \mathcal{I})$ where N is a finite ground set and $\mathcal{I} \subseteq 2^N$ is a collection of independent sets that satisfy the following properties: (i) $\emptyset \in \mathcal{I}$ (non-triviality), (ii) $\forall I \in \mathcal{I}; J \subset I \Rightarrow J \in \mathcal{I}$ (down-closedness), and (iii) $\forall I, J \in \mathcal{I}; |I| < |J| \Rightarrow \exists j \in J \setminus I; I + j \in \mathcal{I}$ (exchange property). The rank function $r_{\mathcal{M}} : 2^N \to \mathbb{Z}_+$ of $\mathcal{M} = (N, \mathcal{I})$ is defined as: $r_{\mathcal{M}}(S) = \max\{|I| : I \subseteq S, I \in \mathcal{I}\}$. $r_{\mathcal{M}}(S)$ is the cardinality of the largest independent set contained in S. The rank of a matroid is $r_{\mathcal{M}}(N)$. It is well-known that the rank function of a matroid is integer-valued, monotone, and *submodular*. A real-valued set function f is submodular iff $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq N$; it is monotone if $f(A) \leq f(B)$ for all $A \subseteq B$. For a matroid $\mathcal{M}$, the convex hull of the characteristic vectors of its independent sets is the matroid polytope $\mathcal{P}(\mathcal{M})$. From the work of Edmonds, $\mathcal{P}(\mathcal{M}) = \text{conv}\{\mathbf{1}_I : I \in \mathcal{I}\} = \{\mathbf{x} \in [0,1]^N : \sum_{i \in S} x_i \leq r_{\mathcal{M}}(S) \text{ for all } S \subseteq N\}$; see [Sch03]. In this paper we will assume that all matroids are loop-less, that is, for all $i \in N$, $r_{\mathcal{M}}(i) = 1$.

Matroids are fundamental objects in combinatorial optimization. They have many applications and rich connections to a variety of areas. The uniform matroid of rank ℓ over n elements, denoted by $\mathcal{U}_{\ell,n}$, is of particular relevance to us. Its independent sets are all subsets of N with cardinality at most ℓ.

Correlation gap: This notion was introduced in [ADSY10] for non-negative real-valued set-functions. Since the definition is technical, and our main interest is in matroid rank functions, we first define it for this special setting.

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Consider a fractional point $\mathbf{x} \in \mathcal{P}(\mathcal{M})$ for some matroid $\mathcal{M}$. The fractional value of this point is $\sum_i x_i$. Suppose we randomly round each i independently with probability x_i to obtain a random set $R(\mathbf{x})$. How does the expected value of the rank of this random set, $\mathbf{E}[r_{\mathcal{M}}(R(\mathbf{x}))]$, compare with $\sum_i x_i$? We are interested in the worst-case ratio between these two quantities. This is the unweighted setting and we can also consider the weighted setting where each element i has a weight y_i and we compare $\sum_i x_i y_i$ with the expected weight of a maximum weight independent set in $R(\mathbf{x})$. We now give a formal definition for a set function.

Definition 1. *For a set function $f : 2^N \to \mathbb{R}_+$, the* correlation gap *is defined as*

$$\kappa(f) = \inf_{\mathbf{x} \in [0,1]^N} \frac{\mathbf{E}[f(R(\mathbf{x}))]}{f^+(\mathbf{x})},$$

where $R(\mathbf{x})$ is a random set containing each element i independently with probability x_i, and

$$f^+(\mathbf{x}) = \max\{\sum_S \alpha_S f(S) : \sum_S \alpha_S \mathbf{1}_S = \mathbf{x}, \sum_S \alpha_S = 1, \alpha_S \geq 0\}$$

is the maximum, over all distributions with expectation $\mathbf{x}$, of the expected value of f.

The quantity $F(\mathbf{x}) = \mathbf{E}[f(R(\mathbf{x}))]$ is the multilinear extension of f [CCPV07], and hence the correlation gap is the worst-case ratio of two continuous extensions of f. See [Dug09] for more on continuous extensions of submodular functions.

We are interested in the correlation gap of the weighted rank function of a matroid $\mathcal{M}$. For a weight vector $y : N \to \mathbb{R}_+$, $r_{\mathbf{y}} : 2^N \to \mathbb{R}_+$ is defined as $r_{\mathbf{y}}(S) = \max_{I \subseteq S, I \in \mathcal{I}} \sum_{i \in I} y_i$. In considering the correlation gap of $r_{\mathbf{y}}$ it suffices to restrict attention to points in the matroid polytope (see Lemma 4.7 in [CVZ14]). With this in mind, the correlation gap of the weighted rank function is then defined as:

$$\inf_{\mathbf{y} \geq 0} \kappa(r_{\mathbf{y}}) = \inf_{\substack{\mathbf{x} \in \mathcal{P}(\mathcal{M}) \\ \mathbf{y} \geq 0}} \frac{\mathbf{E}[\max_{S \subseteq R(\mathbf{x}), S \in \mathcal{I}} \sum_{i \in S} y_i]}{\sum_{i \in N} x_i y_i}$$

Note that the infimum is taken over all weight vectors y. By a relatively simple argument, due to the optimality of the greedy algorithm for maximum independent set in a matroid, one can show that $\inf_{\mathbf{y} \geq 0} \kappa(r_{\mathbf{y}})$ is achieved for the unit weight vector, that is $\inf_{\mathbf{y} \geq 0} \kappa(r_{\mathbf{y}}) = \kappa(r_{\mathcal{M}}) = \inf_{\mathbf{x} \in \mathcal{P}(\mathcal{M})} \frac{\mathbf{E}[r(R(\mathbf{x}))]}{\sum_{i \in N} x_i}$. This was formally shown in [HKLV23]. Hence it suffices to focus on the unweighted case. We highlight that this is only for the sake of simplicity and is not necessary in our analysis. Our proof still holds after substituting all unit rank functions by weighted rank functions.

An important result is that $\kappa(f) \geq (1 - 1/e)$ for any monotone submodular function [CCPV07, Von07, ADSY10]. This implies that $\kappa(r_{\mathcal{M}}) \geq (1 - 1/e)$ for all $\mathcal{M}$; as a function of the number of elements n in the matroid, one can obtain a slightly refined bound of $(1 - (1 - 1/n)^n)$ [CVZ14]. Interestingly, this bound is tight even for the uniform matroid of rank 1; that is $\kappa(\mathcal{U}_{1,n}) = 1 - (1 - 1/n)^n$. The bound improves substantially for uniform matroids of rank ℓ as ℓ grows. Yan [Yan11], and subsequently others [BFGG22, KS23] showed that $\kappa(\mathcal{U}_{\ell,n}) \geq 1 - \frac{\ell^\ell e^{-\ell}}{\ell!}$, and moreover this bound is tight as $n \to \infty$. This latter bound behaves as $(1 - \frac{1}{\sqrt{2\pi\ell}})$ which tends to 1 as $\ell \to \infty$. The setting of $\mathcal{U}_{\ell,n}$ arises two nice applications. The first is in prophet inequalities and mechanism design involving selecting/selling ℓ identical items [Ala14]. Second is in the setting of improved approximation algorithms for maximum multi cover [BFGG22]. Correlation gaps have several applications including the design of contention resolution schemes, mechanism design, and stochastic optimization. A recent paper [HKLV23] gives a nice overview of some of these applications.

ℓ-fold union of a matroid: Although simple, cardinality constraints play an important role in various settings including prophet inequalities, mechanism design, and also in approximation. Partition matroids, which are disjoint union of uniform matroids, further amplify the range of applications. As we saw, the correlation gap improves towards 1 as $\ell \to \infty$. Matroids provide a rich and powerful way to model constraints. However, since they contain the uniform matroid of rank 1 as a special case, the best CR scheme we have is limited to $(1 - 1/e)$. In recent work Husic et al [HKLV23] ask whether this bound can be improved for interesting classes of matroids.

They showed that improvements can be obtained by considering the *girth*[1] γ of the matroid $\mathcal{M}$. For a matroid $\mathcal{M}$ with rank ρ and girth γ, they proved that

$$\kappa(r_{\mathcal{M}}) \geq 1 - 1/e - \frac{e^{-\rho}}{\rho}\left(\sum_{i=0}^{\gamma-2}(\gamma - 1 - i)\left[\binom{\rho}{i}(e-1)^i - \frac{\rho^i}{i!}\right]\right) > (1 - 1/e).$$

As a corollary they prove that the uniform matroid of rank 1 is the worst-case example for matroids.

We ask a related but a different question. Is there a natural generalization of the cardinality constraint with ℓ items to the matroidal setting? Our conceptual contribution is to suggest such a generalization via the classical notion of matroid union that has several fundamental applications in combinatorial optimization [Sch03].

For a matroid $\mathcal{M} = (N, \mathcal{I})$ and integer $\ell \geq 1$ we consider the matroid obtained by taking the ℓ-fold union of $\mathcal{M}$.

Definition 2. (ℓ-fold matroid) *For a matroid $\mathcal{M} = (N, \mathcal{I})$, its ℓ-fold union matroid is defined as*

$$\mathcal{M}^\ell = \underbrace{\mathcal{M} \vee \mathcal{M} \vee \cdots \vee \mathcal{M}}_{\ell \text{ times}} = (N, \mathcal{I}^\ell)$$

where

$$\mathcal{I}^\ell = \{I_1 \cup I_2 \cup \cdots \cup I_\ell \mid I_i \in \mathcal{I}, 1 \leq i \leq \ell\}.$$

Alternatively, a set $A \in \mathcal{I}^\ell$ iff A can be partitioned into at most ℓ independent sets in $\mathcal{I}$. We note that the uniform matroid of rank ℓ can be viewed as the ℓ-fold union of the uniform matroid of rank 1.

Main result: We prove the following theorem.

Theorem 1.1. *For any matroid $\mathcal{M}$ and any integer $\ell \geq 1$,*

$$\kappa(r_{\mathcal{M}^\ell}) \geq 1 - \frac{\ell^\ell e^{-\ell}}{\ell!}.$$

We briefly compare our theorem to that in [HKLV23]. We note that the girth of $\mathcal{M}^\ell$ is at least $\ell+1$, however, the rank ρ of $\mathcal{M}^\ell$ can be arbitrarily large when compared to ℓ. When the rank is large for a fixed girth γ, the gap shown in [HKLV23] converges to $(1 - 1/e)$ and does not provide an improvement, while our bound does not depend on the rank of $\mathcal{M}^\ell$.

Motivation and and some applications: Our motivation came from the intuition that as ℓ increases, the packing constraint imposed by the cardinality constraint becomes loose, and random rounding behaves well. This phenomenon is well-known in several contexts where larger capacity allows better bounds — we refer the reader to the notion of width used in approximating packing integer programs [BKNS12, KRTV18, CQT20], and also improved bounds obtained in various routing problems [BS00, KPP08, HSS11]. It is not quite clear how one makes the constraint imposed by a matroid "loose". We believe that considering $\mathcal{M}^\ell$ via matroid union is one clean approach towards this. As far as we are aware, this question has not been explored previously.

Our focus in this paper is to formulate and prove Theorem 1.1. The applications of correlation gap are well-known and we refer the reader to some past and recent papers for more detailed discussion [Yan11, ADSY10, CVZ14, BFGG22, HKLV23]. Here we mention two of them briefly.

The first one is an application to *contention resolution schemes* (CR schemes). These are a class of randomized rounding schemes that convert a fractional solution x in a polyhedral relaxation $\mathcal{P}$ for a constraint to an integer solution. They were initially formalized [CVZ14] in the context of constrained submodular function maximization and since then they have found several other applications. [CVZ14] established a tight connection between CR schemes for a constraint imposed by an independence family $\mathcal{I} \subseteq N$ and the correlation gap of the weighted rank function corresponding to $\mathcal{I}$. Via this connection, they derived an optimal $(1 - 1/e)$-balanced CR scheme for matroid polytopes. Theorem 1.1 implies that there is a $(1 - \frac{\ell^\ell e^{-\ell}}{\ell!})$-balanced CR scheme for $\mathcal{M}^\ell$. One can compose CR schemes for constraints when considering their intersections [CVZ14], and the scheme for $\mathcal{M}^\ell$ can be used in a black-box fashion to derive further applications.

[1]The girth of a matroid is the smallest size of a circuit (a minimal dependent set) of $\mathcal{M}$.

The preceding application involves matroids as constraints. Another application is when one considers weighted rank functions of matroids as a special case of submodular functions. Consider a submodular function $f : 2^N \to \mathbb{R}_+$ where $f = \sum_{j=1}^{h} g_j$ where each g_j is a weighted rank function of a matroid $\mathcal{M}_j$ on N. Maximum k-Cover is a canonical example of such a function [CCPV07]; we are given n subsets $S_1, \dots, S_n$ of a universe X and the goal is to pick k of these sets to maximize their union. Max k-Cover admits a tight $(1-1/e)$-approximation via several methods. Barman et al [BFGG22] considered the maximum multicover problem where each element can be covered up to ℓ times and showed that it admits an improved $(1 - \frac{\ell^\ell e^{-\ell}}{\ell!})$ approximation. Their result is obtained via the fractional relaxation framework following by pipage/swap rounding; a key difference between this class of functions and general submodular functions is that one can solve an LP relaxation as opposed to using the multilinear relaxation (see [CCPV07, BFGG22]). The key to the result in [BFGG22] is the correlation gap for $\kappa(\mathcal{U}_{\ell,n})$. Via Theorem 1.1, one can obtain a $(1 - \frac{\ell^\ell e^{-\ell}}{\ell!})$ approximation for any submodular function of the form $\sum_{j=1}^{h} g_j$ where each g_j is a weighted rank function of a matroid $\mathcal{M}_j^{\ell_j}$ where $\ell_j \geq \ell$ for each $j \in [h]$.

Our proof of Theorem 1.1 is short and intuitively simple. Via a submodularity inequality first shown in [Von07], it reduces the general case to the setting when $\mathcal{M}^\ell = \mathcal{U}_{\ell,n}$. The original proof of the correlation gap for a submodular set function [CCPV07, Von07] is based on a Poisson clock process, and this approach has been dominant in several subsequent works as well. Our proof is akin to a different proof in [CVZ14]; [CL21] also build upon this latter proof. We had formulated Theorem 1.1 a few years ago following the work of Barman et al [BFGG22]. Our initial attempts at a proof were based on using Karger's matroid base sampling approach [Kar98] while the proof we provide here follows a simpler and direct approach via a reduction to the cardinality case. Since the paper is short, we do not give a separate high-level overview. A reader who is somewhat familiar with prior work on correlation gaps may directly go to the proof of Theorem 3.1 and work backwards to see the utility of the supporting lemmas in Section 3.

2 Preliminaries

We need a characterization of the rank function of $\mathcal{M}^\ell$ in terms of the rank function of $\mathcal{M}$.

Lemma 2.1. (see [Sch03]) *Let $\mathcal{M} = (N, \mathcal{I})$ be a matroid and let $\mathcal{M}^\ell = (N, \mathcal{I}^\ell)$ be its ℓ-fold union. Then, for $S \subseteq N$,*

$$r_{\mathcal{M}^\ell}(S) = \min_{T \subseteq S} \left(|S \setminus T| + \ell \cdot r_{\mathcal{M}}(T) \right).$$

We provide some mathematical results required for proving our main theorem. In the following, we use $\mathrm{Ber}(\cdot)$ and $\mathrm{Poi}(\cdot)$ to denote Bernoulli and Poisson random variables.

Definition 3. (c.f. [SS07]) *Let X and Y be two random variables. X is said to be* smaller that Y in the convex order *(denoted as $X \leq_{\mathrm{cx}} Y$) if*

$$\mathbf{E}[f(X)] \leq \mathbf{E}[f(Y)] \quad \textit{for any convex function } f : \mathbb{R} \to \mathbb{R},$$

which is equivalent to

$$\mathbf{E}[f(X)] \geq \mathbf{E}[f(Y)] \quad \textit{for any concave function } f : \mathbb{R} \to \mathbb{R}.$$

Lemma 2.2. ([BFGG22]) *For any $p \in [0, 1]$, we have*

$$\mathrm{Ber}(p) \leq_{\mathrm{cx}} \mathrm{Poi}(p),$$

Lemma 2.3. (Theorem 3.A.12(d) in [SS07]) *Let $X_1, X_2, \dots, X_m$ be a set of independent random variables and let $Y_1, Y_2, \dots, Y_m$ be another set of independent random variables. If $X_i \leq_{\mathrm{cx}} Y_i$ for $1 \leq i \leq m$, then*

$$\sum_{i=1}^{m} X_i \leq_{\mathrm{cx}} \sum_{i=1}^{m} Y_i.$$

That is, the convex order is closed under convolutions.

We need the following inequality which is probably known in the literature but we give a proof for the sake of completeness.

LEMMA 2.4. *Let $\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}$ be n fractions and $c_1, \dots, c_n$ be n numbers such that $a_i, c_i \in \mathbb{R}_{\geq 0}, b_i \in \mathbb{R}_{>0}$ for $i \in [n]$. Suppose $\frac{a_1}{b_1} \geq \cdots \geq \frac{a_n}{b_n}$ and $c_1 \geq \cdots \geq c_n$, then*

$$\frac{\sum_{i=1}^n c_i a_i}{\sum_{i=1}^n c_i b_i} \geq \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}.$$

Proof. After rearranging terms, we want to prove

$$\sum_{i=1}^n c_i \left(a_i \sum_{j=1}^n b_j - b_i \sum_{j=1}^n a_j \right) \geq 0.$$

Considering the coefficient of the term $a_i b_j$, we have

$$\begin{aligned}
&\sum_{i=1}^n c_i \left(a_i \sum_{j=1}^n b_j - b_i \sum_{j=1}^n a_j \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n a_i b_j (c_i - c_j) \\
&= \sum_{i=1}^n \sum_{j=i+1}^n a_i b_j (c_i - c_j) + a_j b_i (c_j - c_i) \\
&= \sum_{i=1}^n \sum_{j=i+1}^n (c_j - c_i)(a_j b_i - a_i b_j) \geq 0.
\end{aligned}$$

The last inequality follows from $c_j \geq c_i$ and $\frac{a_j}{b_j} \geq \frac{a_i}{b_i}$ for $j > i$. ∎

3 Proof of the Correlation Gap

We start with a lemma that enables us to make parallel copies of elements safely.

LEMMA 3.1. *Let $\mathcal{M} = (N, \mathcal{I})$ a matroid and let $e \in N$. Let $\mathcal{M}' = (N' = N - e + \{e_1, e_2\}, \mathcal{I}')$ be obtained from $\mathcal{M}$ by replacing e with two copies e_1, e_2 and defining $\mathcal{I}'$ as below:*

$$\mathcal{I}' = \{I : e \notin I, I \in \mathcal{I}\} \cup \{I - e + e_1 : e \in I, I \in \mathcal{I}\} \cup \{I - e + e_2 : e \in I, I \in \mathcal{I}\}.$$

Then $\mathcal{M}'$ is a matroid. Further, for any $\boldsymbol{x} \in \mathcal{P}(\mathcal{M}^\ell)$, let $\boldsymbol{x}' \in [0,1]^{N'}$ be such that $x'_i = x_i$ for $i \in N \setminus e$ and $x'_{e_1} + x'_{e_2} = x_e$. We have $\boldsymbol{x}' \in \mathcal{P}(\mathcal{M}'^\ell)$ and

$$\mathbf{E}[r_{\mathcal{M}^\ell}(R(\boldsymbol{x}))] \geq \mathbf{E}[r_{\mathcal{M}'^\ell}(R(\boldsymbol{x}'))].$$

Proof. It is straightforward to check that $\mathcal{M}'$ is a matroid. For ease of notation, let $r(\cdot) = r_{\mathcal{M}^\ell}(\cdot)$ be the rank function of $\mathcal{M}^\ell$ and $r'(\cdot) = r_{\mathcal{M}'^\ell}(\cdot)$ be that of $\mathcal{M}'^\ell$. By Lemma 2.1, it is clear that for any $S \subseteq N - e$, $r(S) = r'(S)$ and $r(S+e) = r'(S+e_1) = r'(S+e_2)$. Hence, it is easy to check that $\mathbf{x}' \in \mathcal{P}(\mathcal{M}'^\ell)$.

Let Y be a random subset of $N - e$ obtained by picking each $e' \in N - e$ independently with probability $x_{e'}$. Let $R(\mathbf{x}) = Y + e$ with probability x_e and $R(\mathbf{x}) = Y$ with probability $1 - x_e$. We analyze $\mathbf{E}[r(R(\mathbf{x}))]$ and $E[r'(R(\mathbf{x}'))]$ conditioned on $Y = T$. Note that $r(T) = r'(T)$ and $r(T+e) = r'(T+e_1) = r'(T+e_2)$. Hence,

$$\mathbf{E}[r(R(\mathbf{x})) \mid Y = T] = (1 - x_e) r'(T) + x_e r'(T + e) = (1 - x_e) r'(T) + x_e r'(T + e_1)$$

and

$$\begin{aligned}
\mathbf{E}[r'(R(\mathbf{x}')) \mid Y = T] = &(1 - x'_{e_1})(1 - x'_{e_2}) r'(T) \\
&+ (x'_{e_1}(1 - x'_{e_2}) + x'_{e_2}(1 - x'_{e_1})) r'(T + e_1) \\
&+ x'_{e_1} x'_{e_2} r'(T + e_1 + e_2).
\end{aligned}$$

Since $x'_{e_1} + x'_{e_2} = x_e$,

$$
\begin{aligned}
&\mathbf{E}[r(R(\mathbf{x})) \mid Y = T] - \mathbf{E}[r'(R(\mathbf{x}')) \mid Y = T] \\
&= x'_{e_1} x'_{e_2} \Big(2r'(T + e_1) - r'(T) - r'(T + e_1 + e_2) \Big) \\
&= x'_{e_1} x'_{e_2} \Big(\big(r'(T + e_1) - r'(T) \big) - \big(r'(T + e_1 + e_2) - r'(T + e_2) \big) \Big).
\end{aligned}
$$

By the submodularity of the rank function, we have

$$r'(T + e_1) - r'(T) \geq r'(T + e_1 + e_2) - r'(T + e_2).$$

Thus, $\mathbf{E}[r(R(\mathbf{x})) \mid Y = T] - \mathbf{E}[r'(R(\mathbf{x}')) \mid Y = T] \geq 0$, and by unconditioning we obtain the desired claim. $\square$

The following lemma plays an important role in the proof.

LEMMA 3.2. ([VON07]) *Let $f : 2^N \to \mathbb{R}_+$ be a monotone submodular function, and let $A_1, \ldots, A_m \subseteq N$. For each $i \in [m]$ independently, sample a random subset $A_i(p_i)$ which contains each element of A_i with probability p_i. Let J be a random subset of $[m]$ containing each element $i \in [m]$ independently with probability p_i. Then*

$$\mathbf{E}\left[f\left(\bigcup_{i \in [m]} A_i(p_i) \right) \right] \geq \mathbf{E}\left[f\left(\bigcup_{i \in J} A_i \right) \right].$$

We will use Lemma 3.2 for $f(\cdot) = r_{\mathcal{M}^\ell}(\cdot)$; recall that the rank functions of matroids are monotone submodular.

For the RHS of the inequality, consider the scenario that $\{A_i\}_{i \in [m]}$ are *disjoint* independent sets in $\mathcal{M}$ and the set A_i appears with probability p_i for $i \in [m]$. Then we can regard each A_i as an element with "weight" $r_{\mathcal{M}^\ell}(A_i) = |A_i|$ and $r_{\mathcal{M}^\ell}(\bigcup_{i \in J} A_i)$ means that we can choose at most ℓ elements among $\{A_i\}_{i \in [m]}$ to form an independent set in $\mathcal{M}^\ell$. This is very similar to the cardinality constraint case.

The preceding observation inspires us to decompose $\mathbf{x} \in \mathcal{P}(\mathcal{M}^\ell)$ into several *disjoint* independent sets in $\mathcal{M}$. Then we can reduce the problem to the case with cardinality constraint ℓ, i.e. rank-ℓ uniform matroid, and follow the proof techniques used in [Yan11, BFGG22, KS23]. We will show in the following formally how the reduction step is performed and provide a clean proof of the cardinality constraint case for the sake of completeness.

LEMMA 3.3. *For a matroid $\mathcal{M} = (N, \mathcal{I})$, let $r(\cdot)$ be the rank funtion of $\mathcal{M}^\ell$ and let $A_1, \ldots, A_m \in \mathcal{I}$ be disjoint independent sets of $\mathcal{M}$. For each $i \in [m]$ independently, sample a random subset $A_i(p_i)$ which contains each element of A_i with probability p_i. If $\sum_{i=1}^m p_i = \ell$, then*

$$\mathbf{E}\left[r\left(\bigcup_{i \in [m]} A_i(p_i) \right) \right] \geq \left(1 - \frac{\ell^\ell e^{-\ell}}{\ell!} \right) \sum_{i=1}^m p_i r(A_i).$$

Proof. By Lemma 3.2, we only need to show

$$\mathbf{E}\left[r\left(\bigcup_{i \in J} A_i \right) \right] \geq \left(1 - \frac{\ell^\ell e^{-\ell}}{\ell!} \right) \sum_{i=1}^m p_i r(A_i).$$

Assume without loss of generality that $r(A_1) \geq r(A_2) \geq \cdots \geq r(A_m)$. We remark that $r(A_i) = |A_i|$ since each A_i is an independent set in $\mathcal{M}$, however we use $r(A_i)$ to indicate that the proof can be easily generalized to the weighted setting without an explicit reduction to the unweighted setting. Let $X_i \sim \mathrm{Ber}(p_i)$ be a random variable indicating whether $i \in J$. Consider the first ℓ elements appearing in J, the corresponding sets are ℓ independent sets in $\mathcal{M}$. Hence the union of them is an independent set in $\mathcal{M}^\ell$. Since they are disjoint, we have

$$\mathbf{E}\left[r\left(\bigcup_{i \in J} A_i \right) \right] \geq \sum_{i=1}^m p_i \Pr\left[\sum_{j=1}^{i-1} X_j < \ell \right] r(A_i).$$

After rearranging terms, we want to prove that

$$\frac{\sum_{i=1}^{m} p_i \Pr\left[\sum_{j=1}^{i-1} X_j < \ell\right] r(A_i)}{\sum_{i=1}^{m} p_i r(A_i)} \geq 1 - \frac{\ell^\ell e^{-\ell}}{\ell!}.$$

We note that we have essentially reduced the problem to the cardinality case. One can appeal to previous work here but we give a self-contained proof.

Since both $\Pr\left[\sum_{j=1}^{i-1} X_j < \ell\right]$ and $r(A_i)$ are decreasing in i, plugging

$$a_i = p_i \Pr\left[\sum_{j=1}^{i-1} X_j < \ell\right], b_i = p_i, c_i = r(A_i)$$

into Lemma 2.4 gives

$$\frac{\sum_{i=1}^{m} p_i \Pr\left[\sum_{j=1}^{i-1} X_j < \ell\right] r(A_i)}{\sum_{i=1}^{m} p_i r(A_i)} \geq \frac{\sum_{i=1}^{m} p_i \Pr\left[\sum_{j=1}^{i-1} X_j < \ell\right]}{\sum_{i=1}^{m} p_i}.$$

Hence it suffices to show that

$$\sum_{i=1}^{m} p_i \Pr\left[\sum_{j=1}^{i-1} X_j < \ell\right] \geq \left(1 - \frac{\ell^\ell e^{-\ell}}{\ell!}\right) \sum_{i=1}^{m} p_i = \left(1 - \frac{\ell^\ell e^{-\ell}}{\ell!}\right) \ell.$$

To see this, note that

$$\begin{aligned}
\sum_{i=1}^{m} p_i \Pr\left[\sum_{j=1}^{i-1} X_j < \ell\right] &= \sum_{i=1}^{m} \sum_{k=0}^{\ell-1} p_i \Pr\left[\sum_{j=1}^{i-1} X_j = k\right] \\
&= \sum_{k=0}^{\ell-1} \sum_{i=1}^{m} p_i \Pr\left[\sum_{j=1}^{i-1} X_j = k\right] \\
&= \sum_{k=1}^{\ell} \Pr\left[\sum_{j=1}^{m} X_j \geq k\right] \\
&= \mathbf{E}\left[\min\left(\sum_{j=1}^{m} X_j, \ell\right)\right].
\end{aligned}$$

For $1 \leq j \leq m$, since $X_j \sim \mathrm{Ber}(p_j)$, we have $X_j \leq_{\mathrm{cx}} \mathrm{Poi}(p_j)$ by Lemma 2.2. Then by Lemma 2.3 we have

$$\sum_{j=1}^{m} X_j \leq_{\mathrm{cx}} \sum_{j=1}^{m} \mathrm{Poi}(p_j) = \mathrm{Poi}\left(\sum_{j=1}^{m} p_j\right) = \mathrm{Poi}(\ell).$$

Since $f(x) = \min(x, \ell)$ is a concave function, by the definition of convex order, we have

$$\begin{aligned}
\mathbf{E}\left[\min\left(\sum_{j=1}^{m} X_j, \ell\right)\right] &\geq \mathbf{E}[\min(\mathrm{Poi}(\ell), \ell)] \\
&= \sum_{k=0}^{\ell} k \cdot \frac{\ell^k e^{-\ell}}{k!} + \sum_{k=\ell+1}^{\infty} \ell \cdot \frac{\ell^k e^{-\ell}}{k!} \\
&= e^{-\ell}\left(\sum_{k=0}^{\ell} \frac{k \cdot \ell^k}{k!} + \sum_{k=\ell+1}^{\infty} \frac{\ell \cdot \ell^k}{k!}\right) \\
&= e^{-\ell}\ell\left(\sum_{k=0}^{\ell-1} \frac{\ell^k}{k!} + \sum_{k=\ell+1}^{\infty} \frac{\ell^k}{k!}\right) \\
&= e^{-\ell}\ell\left(e^{\ell} - \frac{\ell^{\ell}}{\ell!}\right) = \left(1 - \frac{\ell^{\ell} e^{-\ell}}{\ell!}\right)\ell.
\end{aligned}$$

∎

THEOREM 3.1. *For any matroid $\mathcal{M} = (N, \mathcal{I})$ and integer $\ell \geq 1$, $\kappa(r_{\mathcal{M}^{\ell}}) \geq 1 - \frac{\ell^{\ell} e^{-\ell}}{\ell!}$.*

Proof. As discussed in Section 1, it suffices to show that

$$\inf_{\mathbf{x} \in \mathcal{P}(\mathcal{M}^{\ell})} \frac{\mathbf{E}[r_{\mathcal{M}^{\ell}}(R(\mathbf{x}))]}{\sum_{i \in N} x_i} \geq 1 - \frac{\ell^{\ell} e^{-\ell}}{\ell!}.$$

Fix $\mathbf{x} \in \mathcal{P}(\mathcal{M}^{\ell})$. It can be written as the convex combination $\mathbf{x} = \sum_{S \in \mathcal{I}^{\ell}} \alpha_S \mathbf{1}_S$ such that $\sum_{S \in \mathcal{I}^{\ell}} \alpha_S = 1$, $\alpha_S \geq 0$. Each $S \in \mathcal{I}^{\ell}$ can be further decomposed into ℓ *disjoint* independent sets $I_1^S, I_2^S, \ldots, I_{\ell}^S$ in $\mathcal{I}$, that is,

$$\mathbf{1}_S = \sum_{i=1}^{\ell} \mathbf{1}_{I_i^S}.$$

Let $I^S = \{I_1^S, I_2^S, \ldots, I_{\ell}^S\}$. Let $(A_i, \beta_i)_{i \in [m]}$ denote the independent sets in $\mathcal{M}$ where $A_i \in \mathcal{I}$ and

$$\beta_i = \sum_{S \in \mathcal{I}^{\ell}, A_i \in I^S} \alpha_S \leq 1.$$

We have

$$\mathbf{x} = \sum_{S \in \mathcal{I}^{\ell}} \alpha_S \mathbf{1}_S = \sum_{S \in \mathcal{I}^{\ell}} \alpha_S \sum_{i=1}^{\ell} \mathbf{1}_{I_i^S} = \sum_{i=1}^{m} \beta_i \mathbf{1}_{A_i}$$

and

$$\sum_{i=1}^{m} \beta_i = \sum_{i=1}^{m} \sum_{S \in \mathcal{I}^{\ell}, A_i \in I^S} \alpha_S = \sum_{S \in \mathcal{I}^{\ell}} \ell \cdot \alpha_S = \ell.$$

Since $\{A_i\}_{i \in [m]}$ may not be disjoint, in order to apply Lemma 3.3, we build a new matroid $\mathcal{M}'$, $\mathbf{x}' \in \mathcal{P}(\mathcal{M}'^{\ell})$ and $\{A_i', \beta_i\}_{i \in [m]}$ such that $\mathbf{x}' = \sum_{i=1}^{m} \beta_i \mathbf{1}_{A_i'}$. We will ensure that $\{A_i'\}_{i \in [m]}$ are disjoint.

Let $r(\cdot)$ be the rank function of $\mathcal{M}^{\ell}$ and $r'(\cdot)$ be that of $\mathcal{M}'^{\ell}$. At the start, set $A_i = A_i'$ for $i \in [m]$. For an element $e \in N$, suppose it is contained in $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$. Then we make k copies $e_1, e_2, \ldots, e_k$ of e in $\mathcal{M}'$. We assign $x'_{e_j} = \beta_{i_j}$ and let e_j be the replacement of e in A'_{i_j} for $1 \leq j \leq k$. Since $x_e = \sum_{j=1}^{k} x'_{e_j}$, by iteratively using Lemma 3.1, we have $\mathbf{x}' \in \mathcal{P}(\mathcal{M}'^{\ell})$ and

$$\mathbf{E}[r(R(\mathbf{x}))] \geq \mathbf{E}[r'(R(\mathbf{x}'))].$$

After performing this procedure to every element in N, clearly we get disjoint $\{A'_i\}_{i\in[m]}$. Now we are able to apply Lemma 3.3 on $\mathcal{M}', (A'_i, \beta_i)_{i\in[m]}$. Since $\sum_{i=1}^m \beta_i = \ell$, we have

$$\mathbf{E}[r'(R(\mathbf{x}'))] = \mathbf{E}\left[r'\left(\bigcup_{i\in[m]} A'_i(\beta_i)\right)\right] \geq \left(1 - \frac{\ell^\ell e^{-\ell}}{\ell!}\right)\sum_{i=1}^m \beta_i r'(A'_i).$$

Note that $r'(A'_i) = |A'_i|$ here. By the construction of $\mathcal{M}'$ and $\mathbf{x}'$, we see that $\sum_{i\in N} x_i = \sum_{i\in N'} x'_i = \sum_{i=1}^m \beta_i r'(A'_i)$. Therefore

$$\begin{aligned}
\mathbf{E}[r(R(\mathbf{x}))] &\geq \mathbf{E}[r'(R(\mathbf{x}'))] \\
&\geq \left(1 - \frac{\ell^\ell e^{-\ell}}{\ell!}\right)\sum_{i=1}^m \beta_i r'(A'_i) \\
&= \left(1 - \frac{\ell^\ell e^{-\ell}}{\ell!}\right)\sum_{i\in N'} x'_i \\
&= \left(1 - \frac{\ell^\ell e^{-\ell}}{\ell!}\right)\sum_{i\in N} x_i.
\end{aligned}$$

This finishes the proof. ∎

We obtain the following corollary via the connection between correlation gap and CR schemes [CVZ14].

Corollary 3.1. *For any matroid $\mathcal{M}$ and integer $\ell \geq 1$, there exists a $\left(1 - \frac{\ell^\ell e^{-\ell}}{\ell!}\right)$-balanced CR scheme for $\mathcal{M}^\ell$.*

There has been substantial work on prophet inequalities and secretary problems with matroid constraints, and on various special cases of matroids such as $\mathcal{U}_{\ell,n}$ and others. It would be interesting to see which of the results for $\mathcal{U}_{\ell,n}$ can be ported over to matroids of the form $\mathcal{M}^\ell$.

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