CS 573: Algorithmic Game Theory | **Lecture date: Feb 6, 2008**
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1 Network Formation/Design Games

In a game $(N, \langle S_i \rangle, \langle u_i \rangle)$ players are interested in maximizing u_i. In this section we consider games where players incur costs, and hence their goal is to *minimize* costs. These games are represented as $(N, \langle S_i \rangle, \langle c_i \rangle)$ where c_i is the cost function of player i.

Recall that we defined a potential function to prove that a load balancing game had a pure Nash equilibrium. In this section we define a *potential game* as follows.

Definition 1.1 (Potential Game) *A game $G = (N, \langle S_i \rangle, \langle c_i \rangle)$ is a* potential game *if $\exists \Phi : (S = S_1 \times S_2 \times \ldots \times S_n) \to \mathbb{R}$ such that the following is true. If $(s_i, s_{-i}) \in S$ and $c_i(s'_i, s_{-i}) < c_i(s_i, s_{-i})$ then $\Phi\big((s'_i, s_{-i})\big) < \Phi\big((s_i, s_{-i})\big)$.*

Lemma 1.2 *Every finite potential game has a pure Nash equilibrium.*

Proof: Let s be such that $\Phi(s) = \min_{a \in S} \Phi(a)$. It is easy to see s is a Nash equilibrium. □

1.1 Game Definition and Properties

Given a (directed or undirected) graph $G = (V, E)$ and k node pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$, G represents a network. Each pair (s_i, t_i) represents a player i who wants to establish a path between s_i and t_i. Each edge $e \in E$ has a cost $c(e) \geq 0$. We denote by P_i the strategies for player i and define it as follows:

$$P_i = \{p | p \text{ is a path from } s_i \text{ to } t_i\}$$

We also denote by p_i a path in P_i. A user i that aims to use p_i has to pay for the edges in p_i. If multiple players wish to use an edge they share the costs using a *cost-sharing mechanism*, which is an important part of this game. For now, we consider a simple cost-sharing mechanism in which the cost of each edge e is shared equally by all players using it.

We define a *strategy profile* as $\bar{p} = (p_1, p_2, \ldots, p_k)$ where $p_i \in P_i$. Given a strategy profile $\bar{p}$, we represent the number of paths that use an edge e by $l(\bar{p}, e)$. Formally, $l(\bar{p}, e) = |\{i | e \in p_i\}|$. In this setting, the cost function for player i would be: $c_i(\bar{p}) = \sum_{e \in p_i} \frac{c(e)}{l(\bar{p}, e)}$.

This cost-sharing mechanism is called *Shapley* cost-sharing scheme, and the resulting game is called Shapley network design game. Note that different cost-sharing mechanisms define different games. In the next section we study the price of anarchy (PoA) and price of stability (PoS) for this game.

Rosenthal Potential Function. For the Shapley network design game, the following is called the Rosenthal potential function:

$$\Phi(\bar{p}) = \sum_{e\in E} c(e) \sum_{j=1}^{l(e)} \frac{1}{j} = \sum_{e\in E} c(e) H_{l(e)} \quad \text{where } H_k \text{ represents the } k\text{-th Harmonic number}$$

and $l(e)$ is short for $l(\bar{p}, e)$.

Lemma 1.3 *For the Rosenthal potential function we have:*

$$c_i(p_i, p_{-i}) - c_i(q_i, p_{-i}) = \Phi\big((p_i, p_{-i})\big) - \Phi\big((q_i, p_{-i})\big), \ \ \forall i, p_i, q_i, p_{-i}$$

Proof:

$$\begin{aligned} c_i(q_i, p_{-i}) - c_i(p_i, p_{-i}) &= \sum_{e\in(q_i - p_i)} \frac{c(e)}{l(e)+1} - \sum_{e\in(p_i - q_i)} \frac{c(e)}{l(e)} \\ &= \Phi\big((q_i, p_{-i})\big) - \Phi\big((p_i, p_{-i})\big). \end{aligned}$$

□

Thus, Φ is a valid potential function. Therefore, the game is a potential game and has a pure Nash equilibrium. Φ is also an *exact* potential function as defined below.

Definition 1.4 (Exact Potential Function - Exact Potential Game) *For a game* $(N, \langle S_i\rangle, \langle c_i\rangle)$, *function* Φ *is an* exact potential function *if* $\forall i,\ \ c_i(s_i', s_{-i}) - c_i(s_i, s_{-i}) = \Phi\big((s_i', s_{-i})\big) - \Phi\big((s_i, s_{-i})\big)$. *A game for which an exact potential function exists is called an* exact potential game.

1.2 Price of Anarchy and Price of Stability

Theorem 1.5 *If k represents the number of players in the Shapley network design game, $PoA \leq k$ and there exists instances where $PoA = k$.*

Proof: Let $\bar{p} = (p_1, p_2, \ldots, p_k)$ be a Nash equilibrium, and OPT be the total cost in a social optimum. For all i we have $c_i(\bar{p}) \leq \min_{p\in P_i} c(p)$. If this inequality did not hold, player i could pay the entire cost for the cheapest path from s_i to t_i and still pay less compared to the cost of its current path p_i. At the same time, $\min_{p\in P_i} c(p) \leq \text{OPT}$, because OPT has to be at least as much as the total cost of the cheapest path from s_i to t_i. Therefore:

$$\begin{aligned} & c_i(\bar{p}) \leq \min_{p\in P_i} c(p) \leq \text{OPT} \\ \Rightarrow\quad & \sum_i c_i(\bar{p}) \leq k\text{OPT} \\ \Rightarrow\quad & cost(\bar{p}) \leq k\text{OPT} \end{aligned}$$

We provide an example in Figure 1 where PoA is equal to k. In this example if all the players use the edge with cost k then we will have a Nash equilibrium. The total cost in this situation is k. The optimum solution, however, is for all paths to use the edge with cost equal to $1+\varepsilon$. Therefore, PoA $= \frac{k}{1+\varepsilon} \rightarrow k$ as ε approaches zero. $\square$

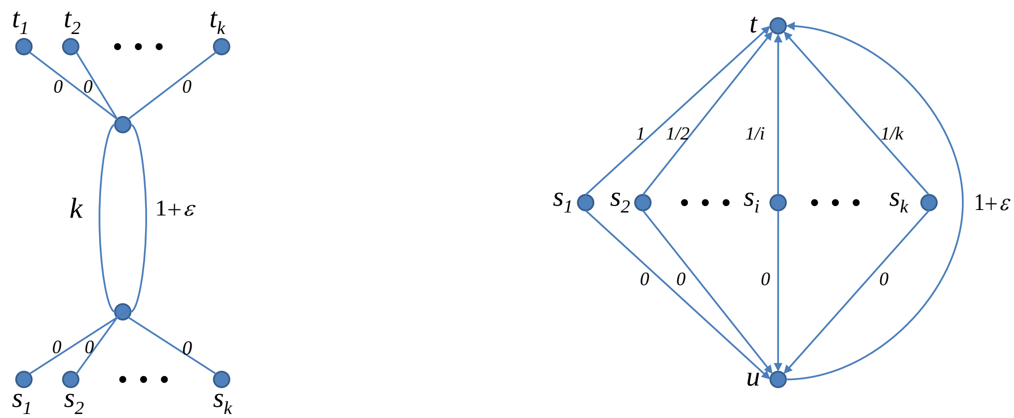

Figure 1: Left picture is an example where PoA is k. Right picture is an example where PoS is H_k.

Theorem 1.6 *If k represents the number of players in the Shapley network design game, PoS $\leq H_k = \Theta(\log k)$, and there exists instances where PoS $= H_k$.*

Proof: First we prove that H_k is an upper bound for PoS. From the definition of Φ it can be checked that:

$$cost(\bar{p}) \leq \Phi(\bar{p}) \leq H_k cost(\bar{p}) \tag{1}$$

Let $p^* = (p_1^*, p_2^*, \ldots, p_k^*)$ be a socially optimum solution. Then we have:

$$\Phi(p^*) \leq H_k cost(p^*) = H_k \text{OPT}$$

Now consider a Nash $\bar{p}$ that minimizes Φ. We have:

$$\Phi(\bar{p}) \leq \Phi(p^*) \leq H_k \text{OPT} \tag{2}$$

$$\text{Equations 1, 2} \Rightarrow cost(\bar{p}) \leq H_k \text{OPT}$$

We provide an example in Figure 1 where PoS is equal to H_k. In this example if each player i uses the edge with cost $\frac{1}{i}$ then we have a Nash equilibrium. Therefore, the cost in this situation is H_k. However, the optimal solution is for all the nodes to use the edge weighted $1+\varepsilon$. Therefore, PoS $= \frac{H_k}{1+\varepsilon} \rightarrow H_k$ as ε approaches zero. Please note that in this example we used a *directed* graph. $\square$

Exercise. Show that the only Nash equilibrium is for each i to use the path $s_i \rightarrow t$.

Theorem 1.7 *For a potential game* $(N, \langle S_i \rangle, \langle c_i \rangle)$*, if* $\forall s \in S$, $\frac{cost(s)}{\alpha} \leq \Phi(s) \leq \beta cost(s)$*, then PoS* $\leq \alpha.\beta$.

Proof: Consider $\bar{s}$ to be a Nash that minimizes Φ. We have:

$$cost(\bar{s}) \leq \alpha.\Phi(\bar{s}) \tag{3}$$

Let s^* be a socially optimum solution. Then we have:

$$\Phi(s^*) \leq \beta.cost(s^*) = \beta.\text{OPT} \tag{4}$$

Since $\Phi(\bar{s}) \leq \Phi(s^*)$, then $\alpha.\Phi(\bar{s}) \leq \alpha.\Phi(s^*)$. From this and Equations 3 and 4 we have:

$$cost(\bar{s}) \leq \alpha.\Phi(\bar{s}) \leq \alpha.\Phi(s^*) \leq \alpha.\beta.cost(s^*) = \alpha.\beta.\text{OPT}$$

□

Open Problem. It is known that for undirected graphs $\frac{12}{7} \leq \text{PoS} \leq H_k$. Obtain a tight bound. In particular, is the PoS a constant? This is unknown even for the single-source setting (all pairs share a source vertex). Note that all our examples so far are essentially for the single-source setting.

2 Congestion Games

The Shapley cost-sharing network design game is a special case of an important class of games called *congestion games*. A congestion game is a four tuple $(N, R, \langle S_i \rangle, \langle c_r \rangle)$ where:

- N is a finite set of n players,
- R is a finite set of resources,
- The set of strategies of the i'th player is $S_i \subseteq 2^R$. Each strategy $s_i \in S_i$ is therefore a subset of the resources of R, in other words $s_i \subseteq R$.
- $c_r : \{1, 2, \ldots, n\} \rightarrow \mathbb{R}^+, \;\; \forall r \in R$, such that $c_r(k) =$ cost of resource r if k players use it.

For $s = (s_1, s_2, \ldots, s_n)$, let $l(s, r) = |\{i | r \in s_i\}|$ which represents the number of players who use resource r. One can think of $l(s, r)$ as the amount of load on r when s is the strategy. Then, the cost of ith player is defined by:

$$c_i(s) = \sum_{r \in s_i} c_r\big(l(s, r)\big)$$

Thus, the cost of a resource r for any player is that uses it is dependent only on the number of players using that resource, and not anything else.

Observation. Shapley cost-sharing game is a congestion game. To see this, R is the set of edges of the given graph and $S_i = P_i$. For an edge e that corresponds to a resource r, we have $c_r(k) = c_e/k$.

Theorem 2.1 (Rosenthal) *Every congestion game is an exact potential game.*

Proof Sketch. Consider the following potential function: $\Phi(s) = \sum_{r \in R} \sum_{j=1}^{l(s,r)} c_r(j)$. Show that Φ is an exact potential for game. □

Theorem 2.2 (Monderer - Shapley) *Every potential game is a congestion game.*

3 Weighted Shapley Game

In this game, different players (*i.e.,* different (s_i, t_i) pairs) have different weights. If $w_i > 0$ represents the weight of player i, the cost-sharing scheme is as follows. For an edge e, each player is charged in proportion to its weight. More formally, for an edge e, the cost for player i is $c(e).\frac{w_i}{w_e}$, where w_e represents the total weight of players that are using edge e.

Theorem 3.1 *For the weighted Shapley game, there exists instances for which no pure Nash equilibrium exists (for an example see [1]).*

Corollary 3.2 *For the weighted Shapley game, no general potential function exists.*

Because of the above, Chen and Roughgarden [1] considered the concept of α-approximate Nash equilibrium as follows. $s = (s_i', s_{-i}) \in S$ is an α-approximate Nash equilibrium if:

$$\forall i, \;\; c_i(s_i', s_{-i}) \geq \frac{1}{\alpha} c_i(s_i, s_{-i}).$$

Intuitively, this condition means that players will not switch strategies if the new cost is not *significantly* better than the current cost (quantified using α).

Theorem 3.3 (Chen - Roughgarden) *For the weighted Shapley game there exists an α-approximate Nash equilibrium of cost $O\left(\frac{\log W}{\alpha}\right)$ times OPT for all $\alpha \in \Omega(\log(w_{\max}))$, where $w_{\max} = \max_i w_i$, and $W = \sum_i w_i$ (assuming that w_is take integer values and the smallest w_i is equal to one).*

Proof Sketch. Proof follows a natural generalization of the potential function for $\bar{p} = (p_1, p_2, \ldots, p_k)$. Let $w_e = \sum_{i: e \in p_i} w_i$. Define $\Phi(\bar{p}) = \sum_e c(e) \log_2(1 + w_e)$. Note that when $w_i = 1, \; \forall i \;\; \Phi(\bar{p}) = \sum_e c(e) \log_2 l(e)$, while Rosenthal's potential was $c(e) H_{l(e)} \approx c(e) \ln(l(e))$. □

In the end, we mention the desired features of cost-sharing schemes:

1. *Budget-balance*: The cost of each edge in the formed network is fully passed on to its users.

2. *Separability*: The cost shares of an edge are completely determined by the set of players that use it, but it may depend on the graph G.

3. *Stability*: For every network design game induced by the cost-sharing protocol, there is at least one pure Nash equilibrium.

4. *Uniformity*: Consider two networks G_1 and G_2, each with the same player set, and two outcomes so that the users of edge $e_1 \in G_1$ and edge $e_2 \in G_2$ are the same subset S in both outcomes. If e_1 and e_2 have equal cost, then the players of S are charged the same cost shares in both outcomes.

It is proven that for any cost-sharing scheme that satisfies the above properties, the PoS is H_k. An interesting, but seemingly unfair, cost-sharing scheme is as follows. The cost for an edge is paid by the least indexed player that uses the edge. It can be proven that in this setting, PoA $\leq O\big(\log^2 k\big)$OPT.

References

[1] Ho-Lin Chen and Tim Roughgarden. Network design with weighted players. In *SPAA*, pages 29–38, 2006.