CS 573: Algorithmic Game Theory	**Lecture date: Feb** $1^{st}, 2008$
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1 Introduction

A proof of Brouwer's fixed point theorem using Sperner's Lemma is discussed below. To recap from a previous lecture, Brouwer's theorem claims that any continuous function that maps any set that is homeomorphic to $[0,1]^d$ (e.g. a closed disc) to itself has a fixed point. We will first prove this generalization and then restrict attention to $[0,1]^d$.

Theorem 1.1 (Brouwer's Fixed Point Theorem) *Let* $f : [0,1]^d \mapsto [0,1]^d$ *be a continuous function. Then there exists a point* $x \in [0,1]^d$ *s.t.* $f(x) = x$*. More generally, this theorem holds for* $f : X \mapsto X$*, where* X *is any set homeomorphic to* $[0,1]^d$*.*

Definition 1.2 (Homeomorphism) *A function* $g : X \mapsto Y$ *where* X *and* Y *are two topological spaces is a homeomorphism if the following three conditions hold:*

- g *is continuous*
- g *is bijective*
- g^{-1} *is continuous*

Note that if $g : X \mapsto Y$ is a homeomorphism, $g^{-1} : Y \mapsto X$ is also a homeomorphism. If there exists such a g mapping two spaces X and Y, we say that X and Y are homeomorphic.

Lemma 1.3 *Let* X *be homeomorphic to* $[0,1]^d$*. Then, the fixed point theorem holds for* X *if it holds for* $[0,1]^d$*.*

Proof: Let $f : X \mapsto X$ be the continuous function being examined. Let g be a homeomorphic function from X to $[0,1]^d$. Define $f' : [0,1]^d \mapsto [0,1]^d$ as $f'(a) = g(f(g^{-1}(a)))$. Since f' is continuous, there exists $a^* \in [0,1]^d$ with $f'(a^*) = a^*$. Let $x^* = g^{-1}(a^*)$. Then, $g(f(x^*)) = g(f(g^{-1}(a^*))) = f'(a^*) = a^* = g(x^*)$. Since, g is bijective, this implies $f(x^*) = x^*$. □

Remark 1.4 *It is also true that any two convex compact full dimensional bodies in* R^d *are homeomorphic to each other.*

2 Sperner's Lemma

We will now prove a combinatorial result to be used for a proof of the fixed point theorem:

Definition 2.1 (Triangulation) *A triangulation,* $\mathcal{T}$ *of a triangle* T *is a set of triangles that exactly covers* T *and mutually intersect only along their edges. Further, let* $V(\mathcal{T})$ *denote the union of the vertices of the triangles covering* T*. (See Figure 1 for an illustration)*

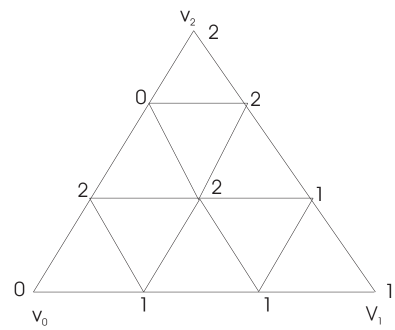

Figure 1: This figure illustrates the triangulation, $\mathcal{T}$ and an example of a valid coloring on $V(\mathcal{T})$. The numbers denote the colors of the vertices. Note that any vertex on the edge v_i, v_j has a color of either i or j. Further, node v_i has color i.

Lemma 2.2 (Sperner's Lemma) *Consider a triangle T with vertices v_0, v_1, v_2. Let $\mathcal{T}$ be a triangulation of T and $V(\mathcal{T})$ denote its set of vertices. Consider any coloring of $V(\mathcal{T})$ with $\{0, 1, 2\}$ such that:*

1. *v_i is colored i. ($i \in \{0, 1, 2\}$)*
2. *If $v \in V(\mathcal{T})$ lies on the line joining v_i and v_j for $i, j \in \{0, 1, 2\}$ the color of v is either i or j.*

Then:

- *There exists at least one triangle in $\mathcal{T}$ whose vertices have all three colors - call such a triangle a multicolored triangle.*
- *Further, the total number of multicolored triangles in $\mathcal{T}$ is odd.*

Proof:

Construct a graph G from $\mathcal{T}$ defined as follows (Refer to Figure 2 for an illustration):

- G has a vertex a_t for each triangle $t \in \mathcal{T}$.
- It has an additional vertex a_0 (corresponds to the outer face).
- Vertices a_t and $a_{t'}$ share an edge iff t and t' share an edge that is colored 0 and 1 on its two incident vertices.
- a_0 is connected to all triangles that have an edge along the line joining v_0 and v_1 with incident vertex colors $0, 1$.

First we note that a_0 has an odd degree. To see this, let α (β) denote the number of $0 - 1$ $(1 - 0)$ color transitions along the line joining v_0 to v_1. Clearly, $\alpha - \beta = 1$. Hence, the degree(a_0) $= \alpha + \beta = 2\alpha - 1$, an odd number.[1] Since the total number of vertices of odd degree in a graph is

[1]Note that this part of the argument is essentially Sperner's Lemma for the case $d = 1$. That is, it says that, if we consider two points and partition the line joining them into multiple segments, and color the points thus obtained with the constraint that the end vertices are 0 and 1, then the total number of multicolored segments along the line will be odd.

always even, the number of vertices with odd degrees *other than* a_0, has to be odd. Now, for any $t \in \mathcal{T}$, degree$(a_t) \in \{0, 1, 2\}$. So, a vertex of odd degree has to be of degree 1. But a vertex can be degree 1 iff it is multicolored. Hence, the number of multicolored triangles is odd.

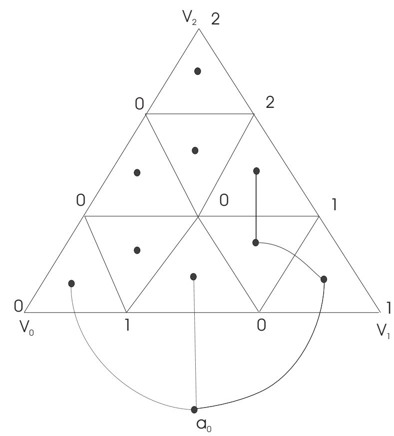

Figure 2: This figure illustrates the graph, G formed from the triangulation $\mathcal{T}$. The dark nodes inside the triangles denote the vertices of G along with a_0. Edges are the (curved) lines between them.

$\square$

3 Fixed Point Theorem

Now consider the fixed point theorem on $[0,1]^d$. For $d = 1$, let $f : [0,1] \mapsto [0,1]$ be continuous. Then the existence of a fixed point for f follows directly from intermediate value theorem. We will now prove the theorem for $d = 2$. It is easy to verify that similar arguments hold for higher dimensions (using an appropriately generalized Sperner's lemma on an n-dimensional simplex).

Theorem 3.1 *Fixed point theorem for* $[0,1]^2$.

Proof: Consider a triangle, T (that is closed). We will prove the theorem on T since it is homeomorphic to $[0,1]^2$. Each $x \in T$ can be uniquely represented as (x_0, x_1, x_2) where $x_i \in [0,1]$ and $x_0 + x_1 + x_2 = 1$ which represents a convex combination of the vertices, $\{v_0, v_1, v_2\}$ (i.e. $x = x_0v_0 + x_1v_1 + x_2v_2$). Note that if x lies on the line joining v_i, v_j, then $x_i + x_j = 1$ and $x_k = 0$ for $k \notin \{i, j\}$. Let us use the notation $x[i]$ to represent x_i, obtained from such a decomposition.

Consider a coloring $g : T \mapsto \{0, 1, 2\}$ defined as follows: Let $y = f(x) \in T$ have the representation (y_0, y_1, y_2). Now, define $g(x) = i$ if i is the smallest coordinate such that $y_i < x_i$. Clearly, this is well defined if f has no fixed point. Further, every point on the line joining v_i, v_j is colored either i or j. (This is because the third coordinate is 0).

Consider a sequence of successively refined triangulations, $< \mathcal{T}_0 (= T), \mathcal{T}_1, \mathcal{T}_2, \ldots >$ such that the size of the triangles decreases by at least a constant factor at each step. Note that the coloring g restricted to the vertices of $\mathcal{T}_i$ satisfies the conditions of Sperner's lemma. So there exists a multicolored triangle $t_i \in \mathcal{T}_i$ for each i. Let its vertices be $\{v_0^i, v_1^i, v_2^i\}$ where v_j^i is the vertex with color j.

Now consider the sequence $< v_0^0, v_0^1, v_0^2, \ldots >$. Since T is compact, there is a convergent subsequence $< v_0^{i_1}, v_0^{i_2}, \ldots >$ that converges to some point $x^* \in T$. Note that since the size of the triangles decreases geometrically, this also implies that the sequences $< v_1^{i_1}, v_1^{i_2}, \ldots >$ and $< v_2^{i_1}, v_2^{i_2}, \ldots >$ both converge to x^*.

Now, by the definition of the coloring, we have $f(v_0^{i_k}[0]) < v_0^{i_k}[0] \; \forall k$. Therefore, by the continuity of f, we conclude by taking the limit that $f(x^*)[0] \leq x^*[0]$. Similarly, we have for each $i \in \{0, 1, 2\}$, $f(x^*)[i] \leq x^*[i]$. But since, $f(x^*)[0] + f(x^*)[1] + f(x^*)[2] = 1 = x^*[0] + x^*[1] + x^*[2]$, the inequalities must in fact be equalities. ie. $f(x^*) = x^*$, leading to a contradiction. Hence, f must have a fixed point. $\square$