CS 573: Algorithmic Game Theory	**Lecture date: 23 April, 2008**
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1 Shapley Value

Given a cooperative cost-sharing game (N, c), we previously considered the core of the game. Each vertex α in the core assigns cost-share α_i to player i such that no coalition $S \subseteq N$ has an incentive to deviate.

The notion of the core is sometimes unsatisfactory for the following reasons:

- The core can be empty
- When the core is non-empty there may be multiple vectors in the core. There needs to be some criterion to decided which vector in the core to choose.

Shapley suggested an axiomatic approach to defining a cost-sharing scheme. Consider a cost-sharing game (N, c). One way to define a cost-sharing scheme for the players is to order them in some way, say $\{1, 2, ..., n\}$, and assign:

$$\alpha_i = c(\{1, 2, ..., i\}) - c(\{1, 2, ..., i-1\}) \tag{1}$$

This cost-sharing scheme depends on the ordering of the players. Thus, it is not anonymous and is not fair. To overcome the lack of anonymity and also the lack of fairness, we can pick a random permutation of the players instead. This gives an *expected* share for each player. To make the cost-sharing scheme deterministic, the average over all permutations can be used.

More precisely, for a permutation σ of players we can define the cost-share of player i in σ as:

$$\phi_i^\sigma(c) = c(\{\sigma^{-1}(1), \sigma^{-1}(2), ..., \sigma^{-1}(j)\}) - c(\{\sigma^{-1}(1), \sigma^{-1}(2), ..., \sigma^{-1}(j-1)\}) \text{ where } \sigma(i) = j \tag{2}$$

Then the Shapley value of i is:

$$\phi_i(c) = \frac{1}{n!} \sum_\sigma \phi^\sigma(i) \tag{3}$$

Note that the Shapley value is defined for any c even if the core does not exist. Also, the Shapley value is, by definition, budget balanced. Unfortunately the Shapley value need not be in the core when the core is non-empty which is stated in the following proposition.

Proposition 1.1 *Even when the core of a cost-sharing game (N, c) is non-empty $\phi(c)$ may not be in the core.*

Proof: See book for an example. □

In the previous lecture it was shown that if c is submodular then for any σ the vector $\phi^\sigma(c)$ is in the core. Then it follows that since the core is convex the Shapley value $\phi(c)$ is in the core. In fact the following holds:

Theorem 1.2 *If c is submodular then $\phi(c)$ is in the core and defines a budget-balanced, cross-monotone, cost-sharing scheme.*

Example: Recall the multicast game when a fixed tree T defined the trees for all $S \subseteq N$. We defined a cross-monotone scheme such that for each $e \in T(S)$ the players in S using e equally shared the cost of e. It can be verified that this is the Shapley value of the game.

Shapley also game another justification for the Shapley value. Consider the following:

Definition 1.3 *Given a set of players N, a "value" ϕ is a function that for any $c : 2^N \to \mathbb{R}$ defines a vector $\phi(c) \in \mathbb{R}_+^N$ with the following properties.*

- *Anonymity: $\phi(c)$ is invarient to permutations of players.*
- *Dummy: If $c(S \cup \{i\}) - c(S) = 0$ for $\forall S \subseteq N \setminus \{i\}$ then $\phi_i(c) = 0$. That is, $\phi_i(c) = 0$ if i does not add cost to any coalition.*
- *Additivity: For every two cost functions c_1 and c_2, $\phi(c_1 + c_2) = \phi(c_2) + \phi(c_2)$.*

Interestingly, Shapley showed the following theorem.

Theorem 1.4 *The Shapley value is the unique value that satisfies anonymity, dummy and additivity.*

2 Market Equilibrium

One aspect of mathematical economics is the notions of markets for goods and equilibrium properties. The study of this area is vast and here we only give a basic background and state some basic results.

In recent years theoretical computer scientists became interested in the algorithmic aspects of market-equilibria, i.e. can the equilibria for a market be computed? Here we discuss three models of markets in increasing order of their complexity.

2.1 Fischer Model

The simplest model is the *Fischer model*:

- There are n goods and m agents/buyers
- each buyer i has an initial endowment of 'money' $e_i > 0$
- There is an initial quantity of good j denoted by $q_j > 0$
- Each player i has a *concave* utility function $u_i : \mathbb{R}_+^n \to \mathbb{R}_+$; i.e. $\forall$ allocation x of goods, $u_i(x)$ is the utility of i for allocation x.

Assumptions:

- Monotonicity: If $y \geq x$ then $u_i(y) \geq u_i(x)$
- Non-Satiability: $\forall x \exists y$ where $y > x$ such that $u_i(y) > u_i(x)$; that is, for every allocation x to player i there is another allocation y which gives player i higher utility than x
- $\forall j$ there is a player i who desires good j

A set of prices $p_1, ..., p_n$ for the goods and an allocation $\bar{x}_1, ..., \bar{x}_m$ where $\bar{x}_{ij}$ is an allocation of good j to buyer i is an equilibrium if:

- The vector $\bar{x}_i$ maximizes $u_i(\bar{x}_i)$ subject to the constraint that $\bar{p}\bar{x} \leq e_i$, i.e. $\bar{x}_i = \max u_i(y)$ and $\bar{p}y \leq e_i$ for $y \geq 0$.
- $\forall j \sum_i x_{ij} = q_j$, i.e. the market clears.

Theorem 2.1 *Market equilibrium always exists if the utility functions u_i are concave, monotone and non-satiable.*

2.2 Exchange Economy

A more general model is that of a *exchange economy*. In this setting there are n goods or commodities and m agents. Each agent i has an initial endowment of goods given by a vector $\bar{w}_i \in \mathbb{R}^n_+$, i.e. w_{ij} represents an amount of good j that agent i is initially endowed. An equilibrium in this setting is a set of prices for the goods $p_1, ..., p_n$ and an allocation of goods $\bar{x}_1, ..., \bar{x}_m$ such that

- $\bar{x}_i$ maximizes $u_i(\bar{x}_i)$ subject to $\bar{p}\bar{x}_i \leq \bar{p}\bar{w}_i$, i.e. $\bar{p}\bar{w}_i$ is the money i makes by selling his initial endowment and then buys $\bar{x}_i$ goods.
- $\forall j$, $\sum_i x_{ij} \leq \sum_i w_{ij}$, i.e. the market clears.

In this setting it can also be shown that an equilibria exists.

2.3 Arrow-Debreu Model

Now we state the most general model of this discussion, the *Arrow-Debreu model*. All the details and assumptions of the model will not be described. The model has the following features:

- n goods, m agents and ℓ firms/manufacturers
- Each agent i has an initial endowment of goods $\bar{w}_i$
- $\forall r$, Each agent i owns a share $d_{i,r}$ of firm r such that $\sum_i d_{i,r} = 1$
- Each firm r has a set $Y_r \in \mathbb{R}^n$ of potential production vectors where $\bar{y} \in Y_r$ implies r can 'produce' $\bar{y}$ such that $y_j < 0$ implies j is being consumed and $y_j > 0$ implies j is being produced.
- Each player has a utility function $u_i : \mathbb{R}^+_n \rightarrow \mathbb{R}^+$

A price vector $\bar{p}$, an allocation $\bar{x_1}, ..., \bar{x_n}$ and a set of production vectors $\bar{y}_1, ..., \bar{y}_\ell$ are in equilibrium if:

- $\forall j$ the total allocation of j to agents and firms cannot exceed $\sum_i w_{ij}$, the initial endowment.
- Each firm r is maximizing profit at current prices i.e. y_r is the production technology that maximizes profit with prices $\bar{p}$
- $\bar{x}_i$ is utility maximizing for i at prices $\bar{p}$ given his wealth.

Note that the wealth of a player i comes from both selling his initial endowment at $\bar{p}$ and profit from his share of the firms.

Theorem 2.2 (Arrow-Debreu) *Under various reasonable and rational conditions an equilibrian exists in the Arrow-Debreu model.*

It is known that in general there could be multiple disconnected equilibria. In some special cases the set of equilibria is a convex set. There is a substantial amount of work on understanding special cases of the discussed models, especially of the exchange economy problem when an equilibrium can be efficiently computed. For the general exchange problem the following in known:

Theorem 2.3 *Finding an equilibrium in the exchange economy model even with explicitly given Leontief utility functions is PPAD-hard*

Thus we do not expect to have a good algorithm.

3 Welfare Theorems

In economics other properties of the equilibrium are also important. An allocation (x^*, y^*) in the Arrow-Debreu model is Pareto optimal if there is no other allocation (x, y) such that $\forall i$ x_i is better than x_i^*.

Theorem 3.1 *(First Welfare Theorem) The equilibria satisfy Pareto optimality*

Theorem 3.2 *(Second Welfare Theorem) Under mild additional conditions, any Pareto optimal feasible allocation (x^*, y^*) can be achieved as an equilibrium with wealth transfer.*