CS 573: Algorithmic Game Theory | **Lecture date: April 18, 2008**
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Contents

1 Group-strategyproof mechanism and cost-sharing schemes

In cost-sharing TU games we assume that all players N wish to get a service and we seek payments in the (approximate) core.

In general the players are price sensitive and may have individual utilities. So they may not wish to participate if the price is too high. However, payment may depend on other players bids. We therefor seek a mechanism that gets truthful bids from the players, decides who to provide service to and how much to charge them.

Unlike the strategic game setting we here also want to protect against coalition. This is a tough requirement, hence we can show the existence of such mechanism only in a limited setting. The ideas of cost-sharing will help and need to be generalized.

1.1 Properties of mechanism

We start with the definition of the setting:

- N is the set of players
- $c : 2^N \to \mathbb{R}^+$ such that $c(S)$ is cost of providing service to $S \subseteq N$.
- v_i is the private value of i for receiving service

A mechanism takes bids $(b_1, \ldots, b_n)$ and decides for each i, $q_i \in \{0, 1\}$ and payment p_i, where we define q_i as

$$q_i = \begin{cases} 1 & \text{if } i \text{ receives service} \\ 0 & \text{if } i \text{ does not receive service} \end{cases}$$

Players are risk neutral, so the utility for player i is $v_i q_i - p_i$.

Moreover we want that a mechanism satisfies the following conditions:

- No Positive Transfers: $p_i \geq 0$ for all i
- Voluntary Participation: $p_i = 0$ if $q_i = 0$, and $p_i \leq b_i$ if $q_i = 1$

- Consumer Sovereignty: for each i there exist some b_i^* such that if $b_i \geq b_i^*$, then i is given the service
- (Approximate) budget-balance: Letting q_i and p_i be functions of bids an $Q(\mathbf{b}) = \{i \in N | q_i(\mathbf{b}) = 1\}$, then for all $\mathbf{b}$ we want

$$\sum_{i \in N} p_i(\mathbf{b}) q_i(\mathbf{b}) \geq \gamma c(Q(\mathbf{b}))$$

- Group-strategyproofnes: Informally this means that it is beneficial for players to be truthful even when collusion with other players is allowed.

  More formally this means, for every $S \subset N$, bids $\mathbf{b}_{-S}$ for $N \backslash S$ and bids $\mathbf{b}_S$ for S, we have:

  If $v_i q_i(\mathbf{b}_S, \mathbf{b}_{-S}) - p_i(\mathbf{b}_S, \mathbf{b}_{-S}) \geq v_i q_i(\mathbf{v}_S, \mathbf{b}_{-S}) - p_i(\mathbf{v}_S, \mathbf{b}_{-S})$ for all $i \in S$, then it holds with equality for every $i \in S$

  In other words, there should *not* be any coalition S such that if S change from $\mathbf{v}_S$ (true values) to $\mathbf{b}_S$, then everyone in S is at least as well-off as before and at least one in S strictly better off.

Note that the standard single-parameter mechanism design setting require truthfulness only for individual players and not coalitions. And recall that even without coalitions we saw that in general truthful mechanisms may not be able to satisfy "budget balance". For instance the public project example.

There are several impossibility results showing that truthfulness and budget-balance are incompatible in most settings. However, we settle for approximate budget-balance and we also consider specific combinatorial problems such as sharing the cost of a multicast tree, where reasonable bounds are possible.

1.2 When can we have a group-strategyproof mechanism?

Definition 1.1 *Let (N, c) denote a cost-sharing game. A* cost-sharing scheme *is a function $\xi : N \times 2^N \rightarrow \mathbb{R}$ such that for every $S \subseteq N$ and every $i \notin S$, $\xi(i, S) = 0$.*

The scheme is γ-budget-balanced if for every set $S \subseteq N$ we have $\gamma c(S) \leq \sum_{i \in S} \xi(i, S) \leq c(S)$.

In other words, a cost-sharing scheme is a collection of cost-shares for each possible set $S \subseteq N$.

Definition 1.2 *A cost-sharing scheme ξ is* cross-monotone *if for all $S, T \subseteq N$ and $i \in S$, $\xi(i, S) \geq \xi(i, S \cup T)$, i.e, cost-share for i decreases as more players are added.*

Proposition 1.3 *Let ξ be a γ-budget-balanced cross-monotonic cost-sharing scheme for a game (N, c). Then $\xi(\cdot, N)$ is in the γ-core of the game.*
Proof: To verify that $\xi(\cdot, N)$ is in the γ-core, we first observe that

$$\gamma c(N) \leq \sum_{i \in N} \xi(i, N) \leq c(N)$$

since ξ is γ-budget balanced. We also need to show that $\forall S \subseteq N$

$$\sum_{i \in S} \xi(i, N) \leq c(S)$$

but by cross-monotonicity

$$\sum_{i \in S} \xi(i, N) \leq \sum_{i \in S} \xi(i, S) \leq c(S)$$

where the last inequality follows from budget-balance of ξ. $\square$

Notice that this proposition implies that cross-monotonicity is a stronger property than core.

1.3 Moulin's group-strategyproof mechanism

We will now see that a cross-monotonic cost-sharing scheme can be used to derive a group-strategyproof mechanism. First we state the mechanism (due to Hervé Moulin):

> Let (N, c) be a cost-sharing game and let ξ be a cross-monotonic cost-sharing scheme. Then the mechanism M_ξ is the following
>
> 1. Take bids $b_1, \ldots, b_n$ from players.
> 2. $S \leftarrow N$
> 3. While $(\exists i \in S$ such that $b_i < \xi(i, S))$ do
>
> $S \leftarrow S \backslash \{i\}$
>
>   end while
> 4. output S as winners and set $p_i = \xi(i, S)$ for all $i \in S$

Clearly the payments charged are less then the bids. We also wish to show that M_ξ is group-strategyproof. To do this, we first prove a useful lemma.

Lemma 1.4 *There is a unique maximal set S such that $\xi(i, S) \leq b_i$ for every $i \in S$, and M_ξ outputs it.*

Proof: Assume that S_1 and S_2 are two distinct maximal sets with the desired property, i.e., $\xi(i, S_1) \leq b_i$ for every $i \in S_1$ and $\xi(i, S_2) \leq b_i$ for every $i \in S_2$.

We argue that $S_1 \cup S_2$ also satisfies the property, thus contradicting the maximality of S_1 and S_2. To see this, consider any $i \in S_1 \cup S_2$. Without loss of generality we can assume $i \in S_1$. Then we have

$$b_i \geq \xi(i, S_1) \geq \xi(i, S_1 \cup S_2)$$

where the second inequality follow from cross-monotonicity of ξ. Hence also $S_1 \cup S_2$ satisfy the property.

Now we argue that M_ξ outputs the unique maximal set S^*, satisfying $\xi(i, S^*) \leq b_i$ for all $i \in S^*$. Suppose not. Let i be the first element from S^* that M_ξ eliminates. Let S be the set in M_ξ at that point. We have $S^* \subseteq S$, since i is the first player to be eliminated.

M_ξ eliminated i because $b_i < \xi(i, S)$. However, by cross-monotonicity, we have $b_i < \xi(i, S) \leq \xi(i, S^*)$, contradicting the definition of S^*. Therefor the set returned by M_ξ must contain S^*. And by maximality of S^*, it cannot contain any other player.

$\square$

Now we prove the following theorem.

Theorem 1.5 *Let (N, c) be a cost-sharing game and let ξ be a γ-budget-balanced cost-sharing scheme for (N, c). Then M_ξ is a γ-budget-balanced group-strategyproof mechanism.*

Proof: Fix a coalition $T \neq N$ and a set of bids b_{-T} for players $N \backslash T$. Let v_i be the true value of each bidder i. Let v_i', $i \in T$, be a set of bids such that for at least one i, $v_i' \neq v_i$.

We wish to show that, whatever the bids of $N \backslash T$ are, if T bids v' instead of v it cannot be that everyone in T is at least as well off as when they bid v and some member of T is strictly better off.

To show this, we consider the bids v_i', $i \in T$ and consider

$$T^+ = \{i \in T \quad | \quad v_i' > v_i\}$$

We claim that $T^+ = \emptyset$. Suppose not, and let $i \in T^+$. We can assume, without loss of generality, that i is a winner when bids are v', for otherwise i does not gain anything.

Now reduce bid of i from v_i' down to v_i while keeping every other bid fixed. There must be some point where i changes from a winner to a loser. Let $b \in (v_i, v_i')$ be the largest value below which i becomes a loser.

Let S_i be the set of winners when $b_i = b$. By Lemma 1.4 we saw that $b = \xi(i, S_i)$, so the payment of i will be $b > v_i$, but this implies that i has a strictly negative utility at bid v_i', and hence i did not benefit. Thus $T^+ = \emptyset$.

Therefore we can assume that $\forall i \in T$, $v_i' \leq v_i$. Let S be the winners with true bids for T, and S' the winners with false bids for T (keeping the bids for $N \backslash T$ fixed). Since $v_i' \leq v_i$ for every $i \in T$, by Lemma 1.4 we know that $S' \subseteq S$. And for every $i \in S'$ payments is $\xi(i, S')$ while it is $\xi(i, S)$ for winners in S. By cross-monotonicity $\xi(i, S') \geq \xi(i, S)$, which implies that no one strictly benefits! $\square$

Thus, to obtain a γ-budget-balanced group-strategyproof mechanism, it is *sufficient* to figure out a γ-budget-balanced cross-monotonic cost-sharing scheme. We note that this is only a sufficient condition and not a necessary condition. There exist group-strategyproof mechanisms for games which do not admit a cost-sharing scheme.

A complete characterization of when a game (N, c) admits a group strategy proof mechanism is not available yet.

1.4 Multicast game

We now consider the Multicast game. We have an undirected graph $G = (V, E)$ with edge costs and a root $r \in V$. N is a set of players residing at vertices of G. Without loss of generality we can assume that they reside at distinct vertices.

There are two versions to consider

1. $c(S)$ is the min-cost Steiner tree connecting S to r.

2. A more realistic version, where there already is a fixed tree T that spans N and $c(S) = \text{cost}(T(S))$, where $T(S)$ is the subtree of T that span S.

For version 1, the core may be empty. However, the $\frac{1}{2}$-budget-balanced core is non-empty, and in fact there is a $\frac{1}{2}$-budget-balanced cross-monotonic cost-sharing scheme. Hence we do get a mechanism.

The above result is based on a primal-dual approximation algorithm and analysis of the Steiner tree problem.

In general there is a strong connection between primal-dual approximation algorithms and cost-sharing schemes.

For version 2 one can show that $c(S)$ is submodular (exercise). And we saw in last class that if c is submodular, then the core is non-empty. In fact we can show the following stronger property.

Theorem 1.6 *If (N, c) is a game where c is submodular, then there is a budget-balanced cost-sharing scheme ξ for c.*

Proof: See textbook.

□

We illustrate the cost-sharing scheme for version 2 of the multicast game, but the same underlying idea works for the general submodular case, although the proof requires more technical machinery to handle arbitrary submodular functions.

Let T be the fixed tree that spans N. To specify a cost-sharing scheme we need to specify $\xi(i, S)$ for each $S \subseteq N$ and $i \in S$.

Since we are looking for a budget-balanced scheme we need to have

$$\sum_{i \in S} \xi(i, S) = cost(T(S))$$

Here is how we do it. Let $P(i)$ be the path from i to r in T. For any edge $e \in T$ let N_e be the set of players in the subtree hanging from e. Then we set

$$\xi(i, S) = \sum_{e \in P(i)} \frac{cost(e)}{|N_e \cap S|}$$

In other words, the cost of any edge $e \in T(S)$ is equally shared by all players whose paths in $T(S)$ contains e.

It is easy to check that

$$\sum_{i \in S} \xi(i, S) = cost(T(S))$$

and that cost-shares are monotone, i.e

$$\xi(i, S) \geq \xi(i, S \cup S') \qquad \forall i, S, S'$$

since the share of i's cost on any edge $e \in P(i)$ can only decrease as more players are added.

Also note that ξ is in a compact form: Given i, S we can easily compute $\xi(i, S)$, but listing all values would require exponential space (in $|N|$).

1.5 Remarks

We have earlier seen that if (N, c) has a γ-budget-balanced cross-monotone cost-sharing scheme, then the γ-core is non-empty. But there are games for which the γ-core is non-empty but the largest γ' such that there is a γ'-budget-balanced cross-monotonic cost-sharing scheme is strictly smaller then γ. The textbook gives a facility location example.

Finally we remark that only a limited class of cost functions c admit cross-monotonic cost-sharing schemes and group-strategyproof mechanism. This is to expected and contrasted with the VCG mechanism which only guarantees truthfulness for individuals.