CS 573: Algorithmic Game Theory	Lecture date: April 11, 2008
Instructor: Chandra Chekuri	Scribe: Hannaneh Hajishirzi

Contents

1 Sponsored Search Auctions

The static model for the keyword auctions is as follows:

- n bidders/advertisers.
- k slots (k is fixed apriori).
- α_{ij} as a click through rate (CTR) of the bidder i if placed in slot j.
- v_i is the value of the bidder i for a click. v_i is independent of the slot position.

In this static model we have the following assumptions:

- Bidders prefer a higher slot to a lower slot (i.e., for every bidder i, $\alpha_{i1} \geq \alpha_{i2} \geq \ldots \geq \alpha_{ik}$) .
- v_i (value of the bidder i) is independent of the slot position.

In particular, allocation (winner determination) problem is assigning bidders to slots to find the optimal allocation which maximizes the utility. In the current setting, it is a single parameter problem given that each bidder has a value v_i per click, and click through rates α_{ij} are public knowledge. In what follows we examine different mechanisms for this allocation problem. We first start with VCG mechanism.

1.1 VCG Mechanism

Let x_{ij} be a $\{0,1\}$ variable that indicates whether slot j is assigned to the bidder i or not. Then, the winner determination problem is to solve the following integer linear program:

$$\begin{aligned}
max \quad & \sum_{i=1}^{n} v_i \sum_{j=1}^{k} \alpha_{ij} x_{ij} & \\
s.t. \quad & \sum_{j=1}^{k} x_{ij} \leq 1 & \forall i \leq n \\
& \sum_{i=1}^{n} x_{ij} \leq 1 & \forall j \leq k \\
& x_{ij} \in \{0,1\} & \forall i \leq n, \forall j \leq k
\end{aligned}$$

Note that we had earlier seen this matching problem, but there v_{ij} could be different from $v_{ij'}$. Here, it is a single parameter problem because $v_{ij} = v_i \alpha_{ij}$ and α_{ij}s are public knowledge. As we saw before, the matching problem has integral value even if we relax the integrality constraints, so we can replace the constraint $x_{ij} \in \{0,1\}$ with $x_{ij} \geq 0$ in the above linear program. We can in fact solve the following linear program:

$$\begin{aligned}
max \quad & \sum_{i=1}^{n} v_i \sum_{j=1}^{k} \alpha_{ij} x_{ij} & \\
s.t. \quad & \sum_{j=1}^{k} x_{ij} \leq 1 & \forall i \leq n \\
& \sum_{i=1}^{n} x_{ij} \leq 1 & \forall j \leq k \\
& x_{ij} \geq 0 & \forall i \leq n, \forall j \leq k
\end{aligned}$$

The dual of this linear program has the following form:

$$min \quad \sum_{j=1}^{k} p_j + \sum_{i=1}^{n} u_i \tag{1}$$

$$s.t. \quad u_i + p_j \geq \alpha_{ij} v_i \qquad \forall i \leq n, \forall j \leq k \tag{2}$$

$$u_i, p_j \geq 0 \qquad \forall i \leq n, \forall j \leq k \tag{3}$$

The intuition is that we set the prices of the slots and let the bidders decide what to do. Finally, we have a Walrassian equilibrium with prices given by optimal dual values.

The Mechanism is truthful and efficient. If CTRs are independent of the bidder (i.e., $\alpha_{ij} = \beta_j$, $\forall i \leq n$) then the assignment can be computed easily. We just assign slots in decreasing order of bids: assign top slot to the top bidder and so on.

1.1.1 Do search engines use VCG?

There are several reasons that search engines do not use VCG for keyword auctions:

1. VCG requires solving a computational problem which needs to be done online for every search and is expensive.

2. VCG can lead to a low revenue for the auctioneer, in this case the search engine. For example, suppose that all the values v_i are the same for all the bidders and α_{ij}s are also the same. Everybody get their slot for the $k+1^{st}$ bid and it is not beneficial for the search engine.

3. VCG can cause problems to the advertisers as illustrated by the following example:

   Example 1.1 *There are 2 bidders and 2 slots given in the following setting:*

   Bidder 1: $\alpha_{11} = 0.8$ $\qquad$ $\alpha_{12} = 0.8$

   Bidder 2: $\alpha_{12} = 0.15$ $\qquad$ $\alpha_{22} = 0.1$

   VCG will assign slot 1 to bidder 2 and slot 1 to bidder 1 even if $v_1 \gg v_2$. *Bidder 1 does not like it because he pays a lots of money and will not go to the top. Furthermore, it is hard to add branding to the model.*

1.2 GFP and GSP Mechanisms

Here, we show some mechanisms used for actual search engines.

Yahoo! used a generalized first price (GFP) auction until 2004.

Definition 1.2 (GFP mechanism) *Let* $b_1, \ldots, b_n$ *be the bids. The GFP mechanism is as follows:*

1. *Sorts bidders according to the bids* $b_1, \ldots, b_n$.

2. *Assigns slots according to the order (assign top slot to the highest bidder and so on).*

3. *Change bidder* i *according to his bid.*

This mechanism is not truthful, but has a Bayesian Nash equilibrium in static setting. Most current search engines now a days use a generalized second price (GSP) auction.

Definition 1.3 (GSP mechanism) *Let* $w_1, \ldots, w_n$ *be the weights on bidders which are static and independent of the bids* $b_1, \ldots, b_n$. *The GSP mechanism is as follows:*

1. *Sort bidders by* $s_i = w_i b_i$.
   (assume) $s_1 \geq s_2 \geq \ldots s_n$.

2. *Allocate slots to bidders* $1 \ldots k$ *in that order (i.e., bidder* i *gets the* i^{th} *slot if* $i \leq k$*).*

3. *Charge* i *the mininum bid he needs to retain his slot (i.e.,* $p_i = \frac{s_{i+1}}{w_i}$*).*

Intuitively, the weights represent some measure that the search engine assigns to the bidder to show some notion of the "quality of the bidder". There are different ways that search engines assign the weights:

- **Overture model** In the Overture model for every $i, w_i = 1$, therefore bidders are ordered according to the bids only. Furthermore, the payment $p_i = b_{i+1}$ for every i.

- **Google model** Google assigns weights based on the CTR at the top slot ($w_i \simeq \alpha_{i1}$). Then, it orders based on $w_i \simeq \alpha_{i1}$. The assumption here is that α_{i1} is static (or slow changing) information and that the bids are varying more quickly. However, the CTR depends on slot assignment and so technically there is feedback. Note that a bidder with low CTR needs to bid higher. This ordering is also called "revenue order" since $s_i = \alpha_{i1}b_i$ is the expected revenue if i is put in slot 1 and there is only one slot.

1.2.1 GSP is not truthful

To understand why GSP is not truthful let us consider a simple example. There are 3 bidders, 2 slots, $w_1 = w_2 = w_3 = 1$, $v_1 > v_2 > v_3$. Assume CTR is bidder independent and $\alpha_{ij} = \mu_j$ and $\mu_1 > \mu_2$. Therefore, bidder 1 gets slot 1 and bidder 2 gets slot 2 in both VCG and GSP. In VCG prices for slots 1 and 2 are $p_1 = (1 - \frac{\mu_2}{\mu_1})v_2$ and $p_2 = v_3$, respectively. In GSP prices for slots 1 and 2 are $p_1 = v_2$ and $p_2 = v_3$, respectively. Since $\mu_1 > \mu_2$ GSP charges bidder 1 more than VCG. Therefore, GSP is not truthful since VCG prices are unique truthful prices and allocations are the same for VCG and GSP.

More concretely, assume that bids of bidders 2 and 3 are fixed. Utility of the bidder 1 when he bids truthfully is $\mu_1(v_1 - p_1)$. However, consider his utility if he bids less than v_2 and gets the second slot: $\mu_2(v_1 - v_3)$. Suppose $v_3 = 0$. Then, for 1 to be truthful we need:

$$\mu_1(v_1 - p_1) \geq \mu_2 v_1 \quad \Rightarrow \quad p_1 \leq (1 - \frac{\mu_2}{\mu_1})v_1$$

But, if $v_2 > (1 - \frac{\mu_2}{\mu_1})v_1$ and GSP charges bidder 1, v_2 then he will bid less than his true value. Even, more concretely:

Example 1.4 *Let $w_1 = w_2 = w_3 = 1$, $v_1 = 200, v_2 = 180, v_3 = 100$ and $\mu_1 = 0.5, \mu_2 = 0.4$. Suppose the bidders bid truthfully, i.e. $b_i = v_i$. Therefore, bidder 1 gets slot 1 and bidder 2 gets slot 2. The expected utility of the bidder 1 is: $0.5(200 - 180) = 10$. Now, suppose bidder 1 bids less than his true value ($b_1 = 110, b_2 = v_2, b_3 = v_3$). Therefore, bidder 1 gets slot 2 and bidder 2 gets slot 1. The expected utility of bidder 1 is $0.4(200 - 100) = 40$ which is greater than when he bids truthfully. In this case, VCG mechanism assigns slot 1 to player 1 and charges her $(1 - \frac{\mu_2}{\mu_1})v_2 = 36$. Furthermore, it assigns slot 2 to player 2 and charges her $v_3 = 100$.*

Therefore, bidding falsely generated more profit, which is not incentive compatible. The intuition is that CTRs for both slots are close, so the bidder 1 does not care which slot to choose. In the above setting, as the CTR for slots 1 and 2 are not that different, bidder 1 bids less and gets the 2^{nd} slot instead of the first one. In VCG mechanism payments are taken into account more cleverly. The payments in VCG are derived by solving the linear program in Equation 1.

We observe that the GSP allocation according to weighted ranking is *monotone*, i.e. increasing the bid of the bidder i and keeping the other bids fixed results in the same slot or a better slot. One question is whether the slot allocation according to weighted ranking can be implemented truthfully or not, i.e. is there a truthful mechanism with GSP allocation rule and a new payment?

From our previous characterization the payment rule is fixed for a truthful implementation of the allocation rule. Recall that,

$$p_i(b, b_{-i}) = bx_i(b, b_{-i}) - \int_0^b x_i(z, b_{-j})dz$$

where, $x_i(b, b_{-i})$ is the CTR of the bidder i with bid b and assuming others are at b_{-i}. (Recall that in previous examples of single parameter problems we considered only the case of $x_i(b, b_{-i})$ being 0 or 1; here we have to be more general). Therefore, one can explicitly compute the payments by employing the above calculation.

Definition 1.5 (Laddered Auction) *Let $w_1, \ldots, w_n$ be the weights on bidders which are static and independent of bids $b_1, \ldots, b_n$. The Laddered auction is as follows:*

- *Allocation is the same as GSP. (Assuming that $S_1 \geq S_2 \ldots \geq S_n$ where $S_i = w_i b_i$).*
- *Payment $p_i = \frac{1}{\alpha_{i,i}} \sum_{j=i}^{k} [(\alpha_{i,j} - \alpha_{i,j+1}) \frac{w_{j+1}}{w_i} b_{j+1}]$*
  or $\alpha_{i,i} p_i = \sum_{j=i}^{k} [(\alpha_{i,j} - \alpha_{i,j+1}) \frac{w_{j+1}}{w_i} b_{j+1}]$
  Note that $\alpha_{i,i} p_i$ is the expected payment of the bidder i in slot i.

We expand the above equation as follows:

$$\begin{aligned} \alpha_{i,i} p_i &= (\alpha_{i,i} - \alpha_{i,i+1}) \frac{w_{i+1}}{w_i} b_{i+1} \\ &+ (\alpha_{i,i} - \alpha_{i,i+2}) \frac{w_{i+2}}{w_i} b_{i+2} \\ &+ \ldots + (\alpha_{i,i} - \alpha_{i,i+k}) \frac{w_{i+k}}{w_i} b_{i+k} \end{aligned}$$

The interpretation of each term in the above equation is as follows: $\frac{w_j}{w_i} b_j$ is the minimum bid of the bidder i to get slot j (for $j \geq i+1$). $(\alpha_{i,j} - \alpha_{i,j+1})$ is the extra click that the bidder i is getting by moving to slot j from slot $j+1$ and the price it should pay is the minimum bid required to get to slot j.

1.3 Equilibrium Issues

The fact that GFP and GSP are not truthful does not mean the revenue is bad. Recall that for a single item GFP admits a Bayesian Nash equilibrium. We saw that for symmetric bidders the revenue equivalence theorem implies that revenue from GFP is the same as any other auction that allocates according to bid order. Note that the equivalence exists under the static model.

In practice the problem is more complicated; In reality, the model is not static, bidders over multiple days may keep changing their bid on the same keyword to find the lowest price. Over time this may result in the loss of revenue for the seller. The Bayesian-Nash equilibrium of GSP is not known even in the symmetric setting. GSP is neither efficient nor it maximizes the revenue. However, one can say the following:

Suppose that CTRs are separable, i.e. for every i and j, $\alpha_{ij} = \beta_i \mu_j$. In other words, CTR depends on the slot and the quality of bidder in the above product form. Then, one can show that the allocation produced by GSP is efficient (maximize social welfare) under a "locally-envy-free" equilibrium.

A locally-envy-free equilibrium is one in which the bidder i has not incentive to switch to $i-1$ or $i+1$ (he may switch to the $i+2$ but not with the immediate neighbors; please refer to the book for more details). Also, please see [1] to see revenue comparison between truthful "laddered" auction and the GSP auction under equilibrium. There exist equilibria (for a fixed valuation) for which the revenue is the same and for which revenues are different.

1.3.1 More issues in sponsored search auctions

Although we looked at static setting, in reality bidders are in a dynamic setting over multiple days stretching to infinity, which makes it quite complicated to analyze. Two phenomenon occurs here: Bid rotation and vindictive bidding.

- **Bid Rotation** Bidders take turns occupying the top slot.
- **Vindictive Bidding** In GSP a bidder can force competitors in a higher slot to pay more by artificially increasing his bid. The advertisers pay based on the click model. One can click on competitors slots and make them pay more to the search engine.

Budget Constraint Issues Each advertiser has a limited budget per day (or multiple days). He/She is bidding over many keywords simultaneously. He/She may run out of budget depending on the search query distribution and clicks.

- **Issues for Advertisers** One issue is that what the advertisers bid on each keyword given query distribution information and CTRs. Besides, queries may consist of multiple keywords.
- **Issues for Search Enging** Different bidders are bidding on different slots of keywords and have budget constraints. Issues include allowing which advertiser to bid, or awarding which advertiser in case of ties to ensure maximum revenue. Consider the following example:

  Example 1.6 *Assume there are two keywords A and B and two bidders 1 and 2.*
  Bidder 1 is interested in keywords A and B
  Bidder 2 is interested in only A
  Suppose query A comes and bidder 1 has little budget and 2 has more budget left. This scenario may cause the search engine to award keyword A to the bidder 2 even if his bid is slightly lower. Save bidder 1 for potential arrival of the keyword B in which bidder 2 is not interested.

References

[1] G. Aggarwal, A. Goel, and R. Motwani. Truthful auctions for pricing search keywords. *ACM conference on Electronic Commerce*, 2006.