CS 573: Algorithmic Game Theory	**Lecture date: 4^{th} April, 2008**
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Bayesian Optimal Mechanism Design

Contents

1 Recap from the last class 1

2 Single parameter, multiple bidders 1

2.1 The model . . . 2

2.2 Derivation . . . 2

2.3 Insights . . . 3

2.4 Some examples . . . 4

3 Auctions with empirical distributions 4

3.1 How good is such an auction? . . . 5

3.2 A randomized auction . . . 5

1 Recap from the last class

- We characterized truthful (incentive compatible) mechanisms for the single parameter setting.
- In particular, we considered the case of a single bidder, for which an auction was designed by introducing a *reserve price*. A reserve price is equivalent to having an extra imaginary bidder in the auction with a fixed bid, equal to the reserve price.
- The optimal auction, in the sense of Myerson for a single bidder, was hence complete characterized by the reserve price set by the auctioneer. It was derived to be

  $$p^* = \arg\max_p p \cdot (1 - F(p)),$$

  where $F(\cdot)$ is the cdf of the valuation of the bidder.
- We introduced another function, $\psi(\cdot)$, called the virtual valuation and defined as

  $$\psi(x) = x - \frac{1 - F(x)}{f(x)}.$$

  Consequently, $p^* = \psi^{-1}(0)$, **if it exists**. Note that ψ need not be positive.

2 Single parameter, multiple bidders

We now derive an optimal mechanism in the Bayesian setting for single parameter bids, with multiple bidders participating in the auction. Since every player's bid is a single real number the terms 'valuation' and 'value' will be used interchangeably. Following is the model.

2.1 The model

Suppose there are n bidders. Assume that each bidder's valuation V_i is a random variable taking values in $[\alpha_i, \beta_i]$ and has an associated distribution F_i supported on $[\alpha_i, \beta_i]$. Furthermore, the valuations of players are assumed to independent. The auctioneer and all other players are assumed to know the collection $F_1, F_2, \ldots, F_n$.

We wish to develop a mechanism that is truthful. In general a mechanism is defined as a collection comprising of an allocation rule and a payment rule. Since payments are completely characterized by allocations, designing a mechanism only amounts to designing an allocation rule. More precisely,

Definition 2.1 *The optimal mechanism design problem is to find an monotone allocation rule* $(w_1, w_2, \ldots, w_n)$ *such the expected sum of payments is maximized. Where the expectation is taken over the joint distribution of* $V_1, \ldots, V_n$.

The following is an assumption we make. Its necessity will be made clear later.

Assumption 2.2 *For each* i *the virtual allocation function* $\psi_i(\cdot)$ *is monotone.*

2.2 Derivation

Suppose $(w_1, \ldots, w_n)$ is a monotone allocation rule. Let P_i be the expected payment of player i. Such a allocation is also truthful, he bids of players are equal to their respective valuations. Let $p_i(v_i, v_{-i})$ denote the payment when the players' valuations are $v_1, \ldots, v_n$. So

$$P_i = \int_{\prod_k [\alpha_k, \beta_k]} p_i(v_i, v_{-i}) f_{1,2,\ldots,n}(v_i, v_{-i}) d(v_i, v_{-i})$$

$$= \int \left(\int_{[\alpha_i, \beta_i]} p_i(x, v_{-i}) f_i(x) \right) f_{-i}(v_{-i}) d(v_{-i})$$

$$\text{Fix } v_{-i}. \text{ Let } \quad P_i(v_{-i}) = \int_{[\alpha_i, \beta_i]} p_i(x, v_{-i}) f_i(x) dx$$

$$\text{Now,} \qquad p_i(x, v_{-i}) = x w_i(x, v_{-i}) - \int_0^x w_i(z, v_{-i}) dz,$$

where $w_i(v_i, v_{-i})$ is an indicator function $w_i : \mathbb{R}^n \to \{0, 1\}$ which takes value 1 if the item is allocated to i, else takes value 0. Therefore

$$P_i(v_{-i}) = \int_{[\alpha_i, \beta_i]} x w_i(x) f_i(x) dx - \int_{[\alpha_i, \beta_i]} f_i(x) \int_0^x w_i(z) dz dx \quad \ldots \text{(dropping the } v_{-i} \text{)}$$

For simplicity we use $w_i(x)$ instead of $w_i(x; v_{-i})$.

$$= \int_{[\alpha_i, \beta_i]} x w_i(x) f_i(x) dx - \int_{[\alpha_i, \beta_i]} w_i(z) \int_z^{\beta_i} f_i(x) dx dz \quad \ldots \text{(change of variables)}$$

$$
\begin{aligned}
&= \int_{[\alpha_i,\beta_i]} x w_i(x) f_i(x) dx - \int_{[\alpha_i,\beta_i]} w_i(z)(1 - F_i(z)) dz \\
&= \int_{[\alpha_i,\beta_i]} w_i(x)(x f_i(x) - (1 - F_i(x))) dx \\
&= \int_{[\alpha_i,\beta_i]} w_i(x) \underbrace{\left(x - \frac{(1 - F_i(x))}{f_i(x)} \right)}_{\psi_i(x)} f_i(x) dx \\
&= \mathbb{E}\left[w_i(V_i) \psi_i(V_i) \right]
\end{aligned}
$$

$$
\begin{aligned}
\text{i.e.} \quad & P_i(v_{-i}) = \mathbb{E}_{V_i}\left[w_i(V_i, V_{-i}) \psi_i(V_i) \right] \\
\text{and} \quad & P_i = \int_{V_{-i}} \left(\int_{[\alpha_i,\beta_i]} w_i(x, v_{-i}) \psi_i(x) f_i(x) dx \right) f_{-i}(v_{-i}) d(v_{-i}) \\
& \quad = \int_{\prod_k [\alpha_k,\beta_k]} \psi_i(x_i) w_i(\bar{x}) f_{1,2,\ldots,n}(\bar{x}) d\bar{x} \qquad \bar{x} = (x_1, x_2, \ldots, x_n)
\end{aligned}
$$

P_i is the revenue from the i^{th} bidder. Finally, total expected revenue of the auctioneer is given by

$$
\sum_{i=1}^{n} P_i = \mathbb{E}\left[\sum_{1}^{n} \psi_i(V_i) w_i(\bar{V}) \right].
$$

2.3 Insights

Thus to maximize expected revenue, the auctioneer has to maximize $\sum_{i=1}^{n} \psi_i(V_i) w_i(\bar{V})$. Consequently the optimal mechanism for this case is simply a VCG applied on $\psi_i(V_i)$ instead of V_i. This justifies the name 'virtual valuation' given to ψ. Formally, the mechanism proceeds as follows.

1. Collect bids $(b_1, b_2, \ldots, b_n)$
2. Computes virtual bids $(\psi_1(b_1), \ldots, \psi_n(b_n))$
3. Solve $\max_{x_i \in \{0,1\}} \sum_{i=1}^{n} \psi_i(b_i) x_i$

As elegant as this looks, there is still a small hitch in this analysis. We had begun our derivation assuming w_i to be monotone valuations. Interpreting the above mechanism as modified VCG requires $\psi_i(b_i)$ to be monotone. Hence the assumption made earlier that ψ_i are monotone. This ensures that the above mechanism is truthful[1].

Note that to compute the optimal allocation, the auctioneer needs knowledge of all F_i. A rather strong requirement. Hence Myerson's theory is most applicable in the symmetric case.

[1] Myerson has shown that in the single item auction case, even in the asymmetric setting, one can get a truthful action even if ψ_i are not monotone. The allocation though is different from this mechanism.

Suppose a seller is trying to sell an item to a single bidder. Then the best that the seller could do is set a reserve price of $\psi^{-1}(0)$. Now if another bidder comes along, then it is seen that the expected revenue of the second price auction (symmetric multiple bidders case below, but without a reserve price) is at least as much revenue as selling to a single bidder. We may say that the seller benefits from competition among his buyers.

2.4 Some examples

We now demonstrate the applicability of the above results with some examples. These are all single item auctions.

1. **Single bidder:** Bid b, compute $\psi(b)$. If $\psi(b) > 0$ allocate the item to the bidder. Else don't sell. This is equivalent to the reserve price being $\psi^{-1}(0)$.

2. **Multiple bidders:** Bidders valuations are i.i.d and drawn from $[0, \omega]$. Let F, ψ be the common cdf and virtual valuation. Compute $\psi(b_1), \psi(b_2), \dots, \psi(b_n)$. Allocate the item to highest bidder if his bid exceeds $p^* = \psi^{-1}(0)$. Payment then is max$\{p^*$, second highest bid$\}$. When the second highest bid exceeds p^*, the payment is the second highest bid since the mechanism is truthful! This mechanism is the same as the second price auction with a reserve price.

3. **Asymmetric bidders:** Bids: $b_1, b_2, \dots, b_n$. Compute $\psi_i(b_i)$ for all i. Allocate item to $i = \arg\max_j \psi_j(b_j)$. i.e. the item needn't get allocated to the highest bidder. The reserve price is $\psi_i^{-1}(0)$ – different for different bidders.

4. k **identical items:** n bidders, each of whom wants a copy of a the k items. $k \leq n$. The VCG mechanism was to award the items to the k highest bidders at the price of the $k + 1^{\text{st}}$ highest bid. But in the Bayesian setting, the allocation is different. If the bidders are symmetric then allocate to those k highest bidders whose bid is above the reserve price $\psi^{-1}(0)$.

5. **Digital goods:** Identical items with $k = n$. i.e. practically unlimited number of items available. In this situation VCG would give the item to each player and charge them 0. But again, due to the Bayesian setting, $\psi^{-1}(0)$ forms the reserve price (for symmetric bidders) and only those players with bids above the reserve price get the item. The revenue is much better than that of a VCG allocation.

3 Auctions with empirical distributions

As discussed above, a significant drawback of the Myerson mechanism is that it requires the knowledge of the cdfs of all bidders. Such information is unavailable in any reasonable setting. This section devotes itself to an 'empirical' Myerson mechanism.

Suppose there are a large number (n) of bidders with bids $b_1, \dots, b_n$. The empirical or statistical cdf is $F_{\bar{b}}(x) = \frac{n_x}{n}$ where $n_x =\#$ bids $\geq x$. The effort is to utilize the empirical cdf in some way to come up with a good auction. Consider the following mechanism.

1. $F_{b_{-i}} = \frac{n_x}{n}$ where $n_x = \#$ bids $\geq x$, excluding i.

2. Simulate the Myerson mechanism using separately for each i using $F_{b_{-i}}$. Such a mechanism is truthful since b_i does not affect $F_{b_{-i}}$.

The above mechanism is in fact a collection of n parallel mechanisms, individually catered for each bidder. There is no clear way of aggregating the outcome of these n into one consistent decision.

Such a problem never occurs with digital goods, as supply is infinite and there is no need for consistency. For digital goods the auction reduces to the following deterministic single price auction (DSOP).

1. Obtain $b_1, \ldots, b_n$.
2. For each i find $F_{b_{-i}}$. Set
$$p_i^* = \arg\max_p p \cdot (1 - F_{b_{-i}}(p))$$

3.1 How good is such an auction?

We find that there is no clear merit for such an auction. Specifically we have the following results.

- **Theorem 3.1** *If bids are i.i.d. random variables with bounded support, then as $n \to \infty$, the DSOP revenue approaches the revenue of the Myerson auction.*
- For arbitrary valuations though, revenue of DSOP can be arbitrarily bad. An example:
  - 10 bids for \$10, 90 bids for \$1. By the above mechanism,
    price offered to \$10 bidder is \$1 $\implies$ revenue = \$10
    price offered to \$1 bidder is \$10 $\implies$ revenue = \$0 The optimal revenue is with a single price set at \$10. Revenue = \$100.
- A negative result:

  Theorem 3.2 *There exist no constants $\gamma, \beta \geq 1$ such that a deterministic anonymous auction of digital goods with valuations in $[1, h]$ can achieve a revenue $\geq \frac{OPT}{\beta} - \frac{h}{\gamma}$, where OPT is the revenue for the best single price auction.*

3.2 A randomized auction

The following is a random sampling version of the empirical Myerson auction (RSOP).

1. Obtain bids $b_1, \ldots, b_n$
2. Partition n into groups N_1, N_2 by putting each bidder randomly into N_1 and N_2 with equal probability.
3. Compute the empirical cdfs F_1 and F_2 using bids in N_1 and N_2 respectively.
4. Run Myerson on N_1 using F_2 and on N_2 using F_1

This mechanism is again truthful and can be applied to digital goods without any hassles of consistency. We have the following powerful positive result.

Theorem 3.3 *For bids in $[1, h]$ the expected revenue of RSOP approaches that of the optimal single price auction as $n \to \infty$.*