CS 573: Algorithmic Game Theory — Lecture date: April 2, 2008
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1 Recap and Notation

In the last class it was seen that reserve prices can increase expected revenue. This prompted the question of an "optimal" mechanism in the more general single parameter setting. A characterization for truthful mechanism in the single parameter setting was described but no proof was given. In this lecture the details will be given.

Let A denote the outcome space, $V_i \subseteq R^A$ denote the valuation space of player i, and $v_i : A \to R$ denotes the valuation of player i for a given outcome. A direct revelation mechanism $< f, p_1, \ldots, p_N >$ consists of $f : V_1 \times V_2 \times \ldots V_N \to A$ the allocation rule and payments $p_i : V_1 \times V_2 \times \ldots V_N \to R$.

2 Characterization in the single parameter case

In the single parameter case V_i is an interval, and v_i a constant function taking a value in interval $[\alpha_i, \beta_i]$. For example, auction of a single item with value v_i known to be in interval V_i and player i gets utility v_i if he wins, otherwise his utility is 0. Another example would be, given a graph $G = (V, E)$ each player i wants a path from s_i to t_i, paths have to be edge disjoint. The valuation of each player i would be a constant v_i and again a player gets utility v_i if his path is routed otherwise he gets 0 utility. The bids of each player b_i is a number in $[\alpha_i, \beta_i]$. Given a mechanism $< f, p_1, \ldots, p_N >$ f is a set of functions $w_1, \ldots, w_N$ such that $w_i(b_1, \ldots, b_N) \in \{0, 1\}$ i.e. given bids $b_1, \ldots, b_N$ the function w_i determines whether player i would win or loose.

Theorem 2.1 *A mechanism that charges losers* 0 *is truthful iff*

1. $\forall i, \forall b_{-i}, w_i(b, b_{-i})$ *is non-decreasing in* b_i.
2. $\forall i, \forall b_{-i}, p_i(b_i, b_{-i}) = 0$ *if* $w_i(b_i, b_{-i}) = 0$ *and* $p_i(b_i, b_{-i}) = argmin_b\{w_i(b, b_{-i}) = 1\}$ *otherwise.*

Another way to state condition 2 above would be $p_i = b_i - \int_0^{b_i} w_i(b, b_{-i})db$.Since the mechanisms considered are randomized so we define $w_i(\bar{b})$ as the probability that player i wins when the bid vector is $\bar{b}$, and $p_i(\bar{b})$ is the expected payment of i on bid vector $\bar{b}$.

A randomized mechanism is truthful in expectation if

$$\forall i, \forall v_i, v_{-i}, E[v_i w_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq E[v_i' w_i(v_i', v_{-i}) - p_i(v_i', v_{-i})]$$

where the expectation is taken on the random choices made by the mechanism.

Theorem 2.2 *A randomized mechanism is truthful in expectation iff*

1. $\forall i, \forall b_{-i} w_i(b, b_{-i})$ *is non-decreasing in* b_i.
2. $\forall i, \forall b_{-i} p_i(b_i, b_{-i}) = 0$ *if* $w_i(b_i, b_{-i}) = 0$ *and* $p_i(b_i, b_{-i}) = b_i w_i(b_i, b_{-i}) - \int_0^{b_i} w(b, b_{-i})db$ *otherwise.*

Note that the previous theorem establishes the deterministic case as well. From this point on we will denote $w_i(b, b_{-i})$ as $w_i(\bar{b})$ for ease.
Proof: If direction: Given player i, valuation v_i and bid b_i the goal is to show that the expected utility u_i of i is maximized when $b_i = v_i$. Expected utility at bid b is given by:

$$u_i(b) = v_i w_i(\bar{b}) - p_i(\bar{b}) = (v_i - b)w_i(\bar{b}) + \int_0^b w_i(\bar{b})db.$$

The above function is minimized at $b = v_i$. This can be shown by differentiating w.r.t. b and equating the derivative to 0 as follows.

$$\dot{u_i(b)} = -w_i(\bar{b}) - (v_i - b)\dot{w_i}(\bar{b}) + w_i(\bar{b}) = (v_i - b)\dot{w_i}(\bar{b}) = 0.$$

Since $w_i(\bar{b})$ is not identically 0 (w_i is not a constant function for all i, b) we have $v_i = b$.
Only if: To prove u_i is truthful in expectation $\implies$ (1) and (2). First the proof of truthful $\implies$ (1) follows. Fix b_{-i} and let $z_1 < z_2$ be two possible bids for player i the aim is to show that $w_i(z_1) \leq w_i(z_2)$. Suppose $v_i = z_1$ then by truthfulness $u_i(z_2, v_i) = u_i(z_2, z_1) \leq u_i(z_2, z_1)$. Hence

$$z_1 w_i(z_2) - p_i(z_2) \leq z_1 w_i(z_1) - p_i(z_1).$$

Similarly if $v_i = z_2$ then bidding z_1 should be of lower expected utility hence

$$z_2 w_i(z_1) - p_i(z_1) \leq z_2 w_i(z_2) - p_i(z_2).$$

Adding the two inequalities followed by some manipulation of terms gives:

$$(z_2 - z_1)(w_i(z_1) - w_i(z_2)) \leq 0.$$

Since $z_2 \geq z_1$ we have $w_i(z_1) \leq w_i(z_2)$. Hence the result. In order to prove truthfulness $\implies$ (2) fix i, b_{-i}. Now u_i is maximized at $b = v_i$ hence $u_i(b) = v_i w_i(\bar{b}) - p_i(\bar{b})$. Hence v_i is a solution of $v_i w_i'(\bar{b}) - p_i'(\bar{b}) = 0$. Therefore $p_i(\bar{b}) = (v_i - b)w_i'(\bar{b})$. This holds for all $v_i \in [\alpha_i, \beta_i]$. Hence $p_i(0) = 0$ and w_i is monotone. In other words p_i satisfies the equation $zw_i'(z) - p_i'(z) = 0$ so $p_i'(z) = zw_i'(z)$ and hence $\int_0^{b_i} p_i'(z)dz = \int zw_i'(z)dz$. Integration by parts gives

$$p_i(b_i, b_{-i}) - p_i(0) = b_i w_i(b_i, b_{-i}) - \int_0^{b_i} w_i(z)dz$$

Since $p_i(0) = 0$ we have $p_i(b_i, b_{-i}) = b_i w_i(b_i, b_{-i}) - \int_0^{b_i} w_i(z)dz$. □

3 Bayesian optimal mechanism design

Consider a single item auction. Suppose only one player participates and the value of the item is known to be in $[\alpha, \beta]$. The question is what can a truthful auction do to extract maximum revenue ? In this case a truthful auction should offer a price p_1 picked randomly to the bidder.

If we use the worst case measure then $\forall v \in [\alpha, \beta]$ the offered price p_1 should be good but it is easy to see that there is no such single price p_1. This issue will be discussed later with worst case auctions. However in the Bayesian case the auction knows that the value of player 1 is drawn from $[\alpha, \beta]$ according to cdf F and hence the question of maximizing the expected value of an auction arises. Hence the question is what price should the auction charge ? Now expected revenue at price p is $pPr(v > p) = p(1 - F(p))$. To find the maximum we need to find the price $p^* = argmax_p(p(1 - F(p)))$. In the case when F is continuous and differentiable we differentiate the expression above and set the derivative to 0. This gives:

$$-pf(p) + (1 - F(p)) = 0, \; p^* = (f(x)/(1 - F(x)))^{-1}$$

where f is the pdf. Define $\phi(x) := x - (1 - F(x))/f(x)$. The value of $\phi^{-1}(0)$ gives the required value of p at which the maximum is attained. In economics terminology $f(x)/(1 - F(x))$ is called the hazard rate of the distribution F. Thus for one bidder we have found an optimal mechanism. It is optimal because the only truthful direct revelation mechanism for one bidder is to define $w_i(\bar{b})$ which is monotone in b. Hence $\forall b \geq p, w_i(b) = 1$ at some price p. It can be verified that the payment is p and hence amongst all mechanisms we have found the one which maximizes expected revenue.