1 Revenue Equivalence Theorem

During last class we studied first and second price auctions in the symmetric Bayesian setting, and we showed that the revenue and expected payments are identical in both auctions at the equilibrium. In fact, this is not just a coincidence but is instead a consequence of the revenue equivalence principle that we will state below.

In general, consider a direct revelation auction A for a single item and suppose that A assigns the item to the highest bidder. Then it can be shown that revenue at equilibrium is the same in all such auctions. Let us consider two further examples.

- **Third price auction:** in third price auction, the item is allocated to the highest bidder who is then charged third highest bid (it is left to the reader as an exercise to prove that such auction is not truthful). Although this auction is not used in practice, it is an interesting exercise to study it.

- **All-pay auction**: again, the item is awarded to the highest bidder. However, each bidder pays his/her bid even if he/she does not win the item. This type of auction is used to model lobbying kind of activities where costs/bids are sunk costs.

Theorem 1.1 (Revenue Equivalence Theorem) *Suppose bidders have independent and identically distributed valuations and are risk neutral. Then any symmetric and increasing equilibrium of a direct revelation auction A that assigns the item to the highest bidder such that the expected payment of bidder with value 0 is 0, yields the same expected revenue.*

The key idea of the theorem is that at equilibrium the revenue depends only on the allocation rule used by the mechanism. As a matter of fact, a more general revenue equivalence theorem states that for any social choice function f implementable by two mechanisms in a Bayesian Nash equilibrium, the payments/revenue will be the same with some normalization assuming losers pay 0 (see the textbook for more details).

Proof: The proof proceeds by deriving properties on the expected payments that do not depend on the specific mechanism being used.

Let $s : [0, w] \to \mathbb{R}$ be a symmetric equilibrium strategy for auction A. We will characterize $m^A(x)$, the expected payment of a fixed bidder (suppose bidder 1 without loss of generality) over v_{-1}, assuming that all other bidders play according to s. By assumption $m^A(0) = 0$. In equilibrium bidder 1 should bid $s(x)$. Suppose he bids some value $s(z)$. We can then derive $m^A(x)$ solving a differential equation as follows.

Let $u^A(z, x)$ be the expected utility of bidder 1 if he bids $s(z)$ assuming his valuation is x. Since the item is allocated to the highest bidder, we can write:

$$\begin{aligned} u^A(z, x) &= x \cdot \Pr[1 \text{ wins with bid } s(z)] - m^A(z) && (1) \\ &= xG(z) - m^A(z), && (2) \end{aligned}$$

where $G(z)$ is the c.d.f. and $g(z)$ is the p.d.f. of $Y_1^{(n-1)}$ as in the previous lecture. Note that $\Pr[1 \text{ wins with bid } s(z)] = G(z)$ since we assumed that the equilibrium strategy is increasing, hence bidder 1 wins if his valuation is greater than all other valuations.

Since bidder 1 chooses z to maximize his expected utility, it must hold:

$$\frac{\partial}{\partial z} u^A(z,x) = xg(z) - \frac{\mathrm{d}}{\mathrm{d}z} m^A(z) = 0 \tag{3}$$

Since furthermore it is optimal to report $z = x$, we obtain:

$$xg(x) - \frac{\mathrm{d}}{\mathrm{d}x} m^A(x) = 0 \tag{4}$$

Finally by solving the differential equation:

$$\begin{aligned} m^A(x) &= m^A(0) + \int_0^x yg(y)\mathrm{d}y && (5)\\ &= 0 + \int_0^x yg(y)\mathrm{d}y && (6)\\ &= G(x)E[Y_i^{(n-1)} | Y_i^{(n-1)} < x] && (7) \end{aligned}$$

Note that $m^A(x)$ does not depend on A, hence the theorem holds. The revenue is equal to $n\int_0^w m^A(x)f(x)\mathrm{d}x$, which is also independent from A. □

It is interesting to note that the theorem can be used to derive equilibria for auctions that would be otherwise difficult to study.

All-pay auction. From the theorem , we know $m^{\mathrm{AP}}(x) = \int_0^x yg(y)\mathrm{d}y$. Furthermore, since every bidder pays his/her bid for an AP auction we have $s(x) = m^{\mathrm{AP}}(x) = \int_0^x yg(y)\mathrm{d}y$ (note that we still need to verify that this is indeed an equilibrium strategy; this is left as an exercise). If F is uniform in $[0,1]$ then:

$$\begin{aligned} s(x) &= \int_0^x yg(y)\mathrm{d}y && (8)\\ &= \int_0^x y(n-1)y^{(n-2)}\mathrm{d}y = (n-1)\int_0^x y^{(n-1)}\mathrm{d}y && (9)\\ &= (n-1)\frac{x^n}{n} = \frac{n-1}{n}x^n && (10) \end{aligned}$$

Third price auction. Again, from the theorem we know $m^{\mathrm{III}}(x) = \int_0^x yg(y)\mathrm{d}y$. We can derive $m^{\mathrm{III}}(x)$ differentially. Let s be the unknown symmetric equilibrium strategy which we assume exists and is increasing. Bidder 1 wins with probability $F_1^{(n-1)}(x)$, and if he/she wins the payment is $s(Y_2^{(n-1)})$ conditioned on the fact that $Y_1^{(n-1)} < x$. We do not report the rest of the computation as it is quite involved, but one can derive that:

$$s(x) = x + \frac{F(x)}{(N-2)f(x)} \tag{11}$$

However, note that the derivation requires that s is increasing, which is true only if $\frac{F(x)}{f(x)}$ in increasing. This in turn is true is F is log-concave (a c.d.f. F is said to be log-concave if $\log F$ is a concave function). In particular, if F is uniform in $[0, 1]$ we obtain:

$$s(x) = x + \frac{x}{N-2} = x(1 + \frac{1}{N-2}), \tag{12}$$

which means that the bid is higher than the valuation.

It is important to realize that the revenue equivalence theorem does not hold if any of the assumptions made by the theorem is violated.

- **Asymmetric bidders:** the theorem only holds for symmetric bidders. In the asymmetric case, there are settings where first price auction yields better revenue than second price auction and viceversa.
- **Risk averse bidders:** the theorem does not hold for risk averse bidders, i.e. if their utility u_i is concave in expected value - payment instead of linear (risk neutral). Intuitively, a risk averse bidder prefers a safe (high probability) small utility than a risky (low probability) high utility. In this setting with symmetric valuations it can be shown than first price auction generates more revenue than second price auction.
- **Budget constraints:** if bidders have budget constraints, meaning they can value the item higher than they can pay for it, the theorem does not hold. Also in this setting first price auction tends to perform better.

Since it is common for one or more of the above assumptions to be violated, in real life auctions sellers often prefer to use a first price auction when interested in revenue maximization.

2 Bayesian Optimal Mechanism Design

In a standard auction, the item is awarded to the highest bidder which has the highest value. Based on the results in the previous section, this implies that all standard auctions have the same revenue for the seller. Hence, the only way for the seller to increase revenue is to change the allocation rule used by the auction.

Example: second price auction with reserve price. In this type of auction, the seller sets a reserve price r for the item. The item is awarded to the highest bidder if the highest bid is greater than r. In this case, the highest bidder is charged the largest of r and the second highest bid. If all bids are less than r, then the item is not sold (resulting in zero revenue for the seller). Showing that this auction is truthful is left as an exercise (suggestion: treat the seller as an additional player).

Let us compute the expected revenue with 2 bidders, F uniform in $[0, 1]$ and reserve price $r = 1/2$. Recall that for a standard auction we showed that revenue is $\frac{n-1}{n+1} = 1/3$. For second price auction with reserve price we have to consider three cases.

1. Both bids are less than $1/2$. This happens with probability $1/4$, and the revenue is 0.
2. Both bids are greater than or equal to $1/2$. The probability is again $1/4$. The expected revenue is the expected value of the lowest bid assuming that both are greater than $1/2$, which is $2/3$.
3. One bid is above $1/2$ and one below $1/2$. The probability is $1/2$, and the revenue is $1/2$.

Hence the expected revenue is:

$$0\frac{1}{4}+\frac{2}{3}\frac{1}{4}+\frac{1}{2}\frac{1}{2}=\frac{5}{12} \tag{13}$$

which is greater than the revenue of 1/3 for a standard auction.

In general, we would like to answer the following question: is there an "optimal" auction, in the sense that it maximizes revenue among all truthful mechanisms? Myerson showed that the answer is positive in the Bayesian setting even when bidders are asymmetric. Note that because of the revelation principle we can stick to direct revelation mechanisms. Furthermore, we will focus on single parameter mechanisms. Recall that in a deterministic single parameter setting, each bidder has a private value $v_i \in V_i$ where V_i is an interval $[\alpha_i, \beta_i]$ on the real line. The bidder either "wins", in which case he obtains utility $v_i - p_i$ where p_i is the payment, or "loses" in which case both the payment and the utility is 0. Let $w_i(b_1, b_2, \ldots, b_N)$ be an indicator variable which determines whether i wins ($w_i = 1$) or loses given bids $b_1, b_2, \ldots, b_N$. Then we showed that a mechanism with 0 payments for losers is truthful iff $\forall i$ and $\forall b_{-i}$:

1. $w_i(b_i, b_{-i})$ is monotone (non-decreasing) in b_i and
2. $p_i(b_i, b_{-i})$ is a critical payment, i.e. it is 0 if b_i is a losing bid and $\text{argmin}_b w_i(b, b_{-i}) = 1$ if b_i is a winning bid.

Note that payments are determined completely by the allocation rule, hence the only flexibility is in the choice of allocation rule.

If the mechanism is randomized then the allocation and payments to a player become random variables. There are two notions of truthfulness for randomized mechanisms. The stronger notion is that of *universal* truthfulness which means that the mechanism is truthful even if the players know the random bits of the mechanism. Such mechanisms are essentially randomizations over deterministic truthful mechanisms. A weaker notion of truthfulness is that of *truthfulness in expectation.* In this setting, let $w_i(b)$ be the probability of i winning under bid vector b. Also let $p_i(b)$ be the expected payment of i under bid b. We assume that bidders are risk neutral, that is, each bidder i is choosing b_i to maximize the expected value of $v_i w_i(b_i, b_{-i}) - p_i(b_i, b_{-i})$.

A mechanism is then said to be truthful in expectation if $\forall i$ and $\forall v_i \in [\alpha_i, \beta_i]$ and $\forall b_{-i}$ it holds:

$$v_i \cdot w_i(v_i, b_{-i}) - p_i(v_i, b_{-i}) \geq v_i \cdot w_i(b, b_{-i}) - p_i(b, b_{-i}) \quad \forall b, \tag{14}$$

i.e. bidding truthfully is expectation maximizing.

Theorem 2.1 *A direct revelation mechanism in which losers pay 0 is truthful in expectation iff for all i and for all b_{-i}:*

1. *$w_i(b, b_i)$ is non-decreasing in b and*
2. *$p_i(b, b_{-i}) = b \cdot w_i(b, b_{-i}) - \int_{\alpha_i}^{b} w_i(z, b_{-i}) \mathrm{d}z$*

The theorem represents a generalization of the previous result, as in the deterministic setting the payments are exactly equal to the critical payments (in particular, the integral is equal to b_i - the critical payment). Furthermore, again the payments are determined completely after w_i has been decided. Starting from the next lecture we will study how we can manipulate w_i to obtain revenue maximizing mechanisms.