CS 573: Algorithmic Game Theory	Lecture date: March 26th, 2008
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Contents

Auctions in the Bayesian setting

We have seen mechanism design in the "worst cass" setting. That is the dominant strategy implementations we sought. Although dominant strategy implementation is ideal, it is a very strong requirement.

Economists also favor Bayesian-Nash implementation since it allows for more mechanism as well as capturing more realistically "prior" information that players have about the valuation of other players.

We will study some types of auctions in the Bayesian setting which allows us to show:

- The revenue equivalence theorem
- Optimal mechanism design

both due to Roger Myerson (also done independently by Riley and Samuelson) who won the Nobel prize in 2007.

1 Single item auction

1.1 Setting

- N players
- Player i's valuation is in $[0, \omega]$ and there is a probility distribution on $[0, \omega]$ that gives the value of i
- The valuation of players are independent
- All players know the distribution of all other players, but not the actual realization of the value. We will mostly focus our attention on the "symmetric" case where the distribution of all the players are the same. This assumption is reasonable in a setting with a large number of players.

Let $X_1, \ldots, X_n$ denote the independent identically distributed random variables that define the value of the players. We use F and f to denote the cumulative distribution function (*cdf*) and the density function repectively.

Assumption 1.1 *F is continuous and differentiable*

Note that

$$\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$$

In this simple setting we can ask several interesting questions:

1. Does the first price auction have an equilibrium (Bayesian Nash)?
2. How does the first price auction compare to that of the second price auction in terms of revenue to the sellers and the expected payment of each player?
3. Is there an "optimal" auction in terms of maximizing the revenue for the sellers?

Quick review of some useful facts from probability that we will need: $X_1, \ldots, X_n$ are independent, identically distributed variables on $[0, \omega]$. *cdf* is $F(x)$, *pdf* is $F'(x) = f(x)$.

Given such variables we can define $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ as the order statistics. $Y_k^{(n)}$ is the random variable that desribes the k'th largest value among $X_1, \ldots, X_n$. Thus $Y_1^{(n)}$ is the largest and $Y_2^{(n)}$ is the second largest, etc.

$$F_1(x) = cdf(Y_1^{(n)})$$

and

$$F_1(x) = \big(F(x)\big)^n$$

therefore,

$$f_1(x) = n\big(F(n)\big)^{(n-1)}$$

What about $F_2(x)$? $F_2(x) =$ probability that second largest among $X_1, \ldots, X_n$ that is less then x. When does this happen?

1. All $X_i \leq x$ Or
2. Exactly only one $X_i > x$ and the rest $\leq x$

$$F_2(x) = \big(F(x)\big)^n + n\big(1 - F(x)\big)\big(F(x)\big)^{(n-1)}$$

and

$$f_2(x) = F_2'(x)$$

1.2 Second Price Auction

We have already seen that a dominant strategy in second price auction is to be truthfully bid the true value. Recall that a strategy of a player is a function $s_i : [0, \omega] \to [0, \omega]$ that maps a value to a bid. For instance $b_i = s_i(v_i)$, where v_i is the true value of i realized from X_i.

In second price auction $s_i(v_i) = v_i$ is a dominant strategy. What is expected revenue of the seller? It is simply $\mathrm{E}[\mathrm{Y}_2(n)]$.

Example 1.1 *Say each player's value is distributed uniformly in* $[0, 1]$*, therefore*

$$
\begin{aligned}
f(x) &= 1 && (1)\\
F(x) &= x && (2)\\
F_2(x) &= \big(F(x)\big)^n + n\big(1 - F(x)\big)\big(F(x)\big)^{(n-1)} \\
&= x^n + n(1-x)x^{(n-1)} && (3)\\
& && (4)\\
\Rightarrow E[\mathrm{Y}_2] &= \int_0^1 x \cdot f_2(x)\mathrm{d}x \\
&= \frac{n-1}{n+1} && (5)
\end{aligned}
$$

Since each player is equally likely to win, the expected payment of player i is $\frac{1}{n}$ times expected revenue, which equals to $\frac{1}{n} \cdot E[\mathrm{Y}_2^{(n)}]$.

Another way to consider the payment of player i is as follows: Let $m(x)$ be payment of i with value x,

$$
\begin{aligned}
m(x) &= Pr[\ i \text{ wins at } x\ ] \cdot E[\text{ second highest bid value } \mid \text{ given it is } \leq x] && (6)\\
&= Pr[\mathrm{Y}_1^{(n-1)} < x] \cdot E[\mathrm{Y}_1^{(n-1)} | \mathrm{Y}_1^{(n-1)} < x] \\
&= F_1^{(n-1)}(x) \cdot E[\mathrm{Y}_1^{(n-1)} | \mathrm{Y}_1^{(n-1)} < x] \\
& && (7)
\end{aligned}
$$

1.3 First Price Auction

Suppose the auctions is a first price auction.

Questions 1.1 *Is there a* Bayesian-Nash *equilibrium? If there is, what are the equilibrious bidding strategies?*

We will show existence of a symmetric equalibrium with the same bidding strategy for each player. We derive it in a heuristic fashion as follows: Let $s : [0, \omega] \to [0, \omega]$ be an equilibrium bidding strategy. We expect $s(x)$ to be less than x. We also expect that $s(x)$ is increasing as a function of x and further

$$s(0) = 0 \qquad \textit{and} \qquad s(x) \leq \omega \qquad \forall x \in [0, \omega]$$

Assume that s is differentiable. Fix a bidder say 1. Suppose x is his value, what should $b(x)$, his bid be?

Player 1 gets utility only if he wins and the bids of the other players will be $s(X_2), s(X_3), \ldots, s(X_n))$ and since we "assumed" that s is increasing, the highest bid that player 1 needs to worry about is

$$\max_{i\neq 1} s(X_i) = s(\max_{i\neq 1} X_i) \tag{8}$$

$$\max_{i\neq 1} X_i = Y_1^{(n-1)} \tag{9}$$

For ease of notation, let us call $Y_1^{(n-1)}$ as Z. Let G and g be *cdf* and *pdf* of Z. Thus player 1 with value x expects a highest bid with distribution $s(Z)$. Player 1 will win if $b(x) > s(Z)$ in which case his utility would be $x - b(x)$.

Thus the expected utility of player i is

$$\begin{aligned}\big(x - b(x)\big) \cdot Pr[\ x \text{ wins }\ ] &= \big(x - b(x)\big) \cdot Pr[s(Z) < b(x)] \\ &= \big(x - b(x)\big) \cdot Pr[Z < s^{-1}\big(b(x)\big)] \\ &= \big(x - b(x)\big) \cdot G[s^{-1}\big(b(x)\big)]\end{aligned} \tag{10}$$

Thus player 1 chooses to maximize his expected utility. To make this easier, let us fix x and b. Expected utility is

$$(x - b)G\big(s^{-1}(b)\big)$$

To maximize, this b should satisfy

$$(x - b)G'\big(s^{-1}(b)\big) \cdot \frac{\mathrm{d}}{\mathrm{d}b} s^{-1}(b) - G\big(s^{-1}(b)\big) = 0 \tag{11}$$

$$\frac{(x - b)G'\big(s^{-1}(b)\big)}{s'\big(s^{-1}(b)\big)} - G\big(s^{-1}(b)\big) = 0 \tag{12}$$

Since $G' = g$, we can update above equation as

$$\frac{(x - b)g\big(s^{-1}(b)\big)}{s'\big(s^{-1}(b)\big)} - G\big(s^{-1}(b)\big) = 0 \tag{13}$$

Since we postulated a symmetric equilibrium, we should have

$$b(x) = s(x) \tag{14}$$

$$\Rightarrow s^{-1}(b) = x \tag{15}$$

Therefore (13) becomes

$$\frac{\big(x - s(x)\big)g(x)}{s'(x)} - G(x) = 0 \tag{16}$$

$$\Rightarrow xg(x) = s(x)g(x) + s'(x)G(x) \tag{17}$$

Since $G' = g$, (17) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(s(x)G(x)\big) = xg(x) \tag{18}$$

Since $s(0) = 0$, we have

$$s(x) = \frac{1}{G(x)} \int_0^x yg(y)\mathrm{d}y \tag{19}$$

We make the observation that

$$s(x) = E[Z|Z < x] \tag{20}$$

At a given value x, player 1 will win only if $Z \leq x$. The expected value of Z conditioned on this is $E[Z|Z < x]$. Thus player 1 wants to pay just above this to continue to win but keep his payment as low as possible (to maximize this utility).

We have heuristically derived the symmetirc equilibrious strategy. We now formally prove that it is indeed an equilibrium.

Lemma 1.2

$$s(x) = E[Y_1^{(n-1)}|Y_1^{(n-1)} < x] \tag{21}$$

is a symmetric equilibrium strategy in a first price auction.

Proof: To show this, we assume that all players except player 1 use s and argue that it is optimal for player 1 also to use s.

Note that s maps $[0, \omega]$ to $[0, s(\omega)]$. Player 1 will never bid greater than $s(\omega)$, since he can win the auction at $s(\omega)$ always and hence we can assume that player 1 also bids from $[0, s(\omega)]$.

Fix any $x \in [0, \omega]$. We want to show that player 1 maximizes his utility by bidding $s(x)$. Suppose he bids some other value $b \in [0, s(\omega)]$. Since s is continuous, there is a z, so that $s(z) = b$. Let

$u(x,t)$ be the expected utility of player 1 to bid t when his value is x. Then

$$\begin{aligned}
u(x,t) &= (x-t)\cdot Pr[\text{ 1 wins at } t \text{ }] && (22)\\
&= (x-t)G\big(s^{-1}(t)\big)\\
\Rightarrow u(x,b) &= \big(x-s(z)\big)G(z) && (23)\\
&= G(z)x - G(z)s(z)\\
&= G(z)x - \int_0^z yg(y)\mathrm{d}y\\
&= G(z)x - G(z)\cdot\frac{1}{G(z)}\int_0^z yg(y)\mathrm{d}y\\
&= G(z)x - \int_0^z yg(y)\mathrm{d}y\\
&= G(z)x - G(z)z + \int_0^z G(y)\mathrm{d}y\\
&= G(z)(x-z) + \int_0^z G(y)\mathrm{d}y\\
u\big(x,s(x)\big) &= \int_0^x G(y)\mathrm{d}y && (24)\\
\Rightarrow u\big(x,s(x)\big) - u\big(x,s(z)\big) &= G(z)(z-x) + \int_x^z G(y)\mathrm{d}y && (25)\\
&\geq 0 \qquad \text{(Since G is increasing :} \geq \text{ holds for } z\geq x \text{ or } z\leq x.)
\end{aligned}$$

We can write

$$\begin{aligned}
s(x) &= \frac{1}{G(x)}\int_0^x yg(y)\mathrm{d}y && (26)\\
&= \frac{1}{G(x)}\big[x\big(G(x)\big) - \int_0^x G(y)\mathrm{d}y\big]\\
&= x - \int_0^x \frac{G(y)\mathrm{d}y}{G(x)}
\end{aligned}$$

Thus $s(x) < x$.

$$\begin{aligned}
G(y) &= F(y)^{(n-1)} && (27)\\
\frac{G(y)}{G(x)} &= (\frac{F(y)}{F(x)})^{(n-1)} \rightarrow 0 \text{ as } n\rightarrow\infty && (28)
\end{aligned}$$

Hence $s(x)\rightarrow x$ as $n\rightarrow\infty$ for fixed F. □

What about expected payment of each player and expected revenue to the seller? It is easier to analyze the expected payment of players.

Let $m(x)$ be expected payment of palyer 1 if x is his value.

$$\begin{aligned}
m(x) &= Pr[\text{1 wins}]\cdot s(x) && (29)\\
&= G(x)s(s)\\
&= G(x) - E[Y_1^{(n-1)}|Y_1^{(n-1)} < x]
\end{aligned}$$

Same as the expected payment of players in second price auction! Therefore, the revenue is same as well.

Example 1.2 *Uniform distribution on* $[0, 1]$,

$$F(x) = x \tag{30}$$

$$G(x) = x^{(n-1)} \tag{31}$$

$$\begin{aligned} \Rightarrow s(x) &= x - \int_0^x \frac{G(y)\mathrm{d}y}{G(x)} \\ &= x - \frac{1}{x^{(n-1)}} \cdot \int_0^x y^{(n-1)\mathrm{d}y} \\ &= x - \frac{x}{n} \\ &= x(\frac{n-1}{n}) \end{aligned} \tag{32}$$

So player i bids slightly under the true valuation.