CS 573: Algorithmic Game Theory	**Lecture date: March 14, 2008**
Instructor: Chandra Chekuri	**Scribe: James Lai**

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1 Bayesian Nash Equilibrium

Our previous definition of mechanisms is suitable if there exist dominant strategy equilibriums. However, this may be too strong a requirement. For example, there is no dominant strategy equilibrium for a fixed price auction. Also, economists favor the Bayesian approach to knowledge in which instead of the strict private value model, they assume that there is prior distributional knowledge about other people's valuations.

Definition 1.1 (Bayesian Game) *A Bayesian game on a set of N players has the following:*

- *every $i \in N$ has a typespace T_i*
- *every $i \in N$ has a set of possible actions X_i*
- *every $i \in N$ has a probability distribution D_i on T_i*
- *every $i \in N$ has a utility function $u_i : T_1 \times X_1 \times X_2 \times ... \times X_n \rightarrow \mathbb{R}$*

We assume that

- All players know $D_1, ..., D_n$
- The type t_i of i is the outcome drawn from D_i independently of other players
- Players are risk neutral expectation maximizers

Definition 1.2 (Bayesian Nash Equilibrium) *A strategy for i is a function $s_i : T_i \rightarrow X_i$. A profile of strategies $(s_1, ..., s_n)$ is a Bayesian Nash Equilibrium if and only if for all i, for all $t_1, ..., t_n$, and for all $x'_i \in X_i$*

$$E_{D_{-i}}[u_i(t_i, s_i(t_i), s_{-i}(t_{-i})] \geq E_{D_{-i}}[u_i(t_i, x'_i, s_{-i}(t_{-i})]$$

Note that the expectation is over the random choices of t_{-i} from D_{-i}

We can define mechanisms as before except now the D_i is public information. A mechanism implements a social choice function f if for all $t_1, ..., t_n$, there is a Bayesian Nash equilibrium $(s_1, ..., s_n)$ such that

$$a(s_1(t_1), ..., s_n(t_n)) = f(t_1, ..., t_n)$$

Remark 1.3 *It can be shown that a direct revleation mechanism exists whenever a mechanism exists (both with respect to Bayesian Nash equilibrium)*

2 Mechanisms Without Money

Recall that the Gibbard-Satterthwaite theorem precludes incentive compatible mechanisms for social choice functions f if $|A| \geq 3$ and f is onto, where A is the outcome space. However, the theorem applies because players have arbitrary preferences profiles over A. One way to circumvent the theorem is to use payments and we have described mechanisms such as VCG and explored various settings under which mechanisms can be designed. We give a few examples of mechanisms without money.

2.1 Single-Peaked Preferences

Definition 2.1 (Single-peaked preference relation) *Let A be an interval on $\mathbb{R}$. Without loss of generality, let $A = [0, 1]$. A preference relation $\prec$ on A is said to be single-peaked if there is a $p \in [0, 1]$ such that*

1. *for all $x \neq p$,* $\quad x \prec p$.
2. *for all $x \neq p$, and $\lambda \in [0, 1)$,* $\quad x \prec (\lambda x + (1 - \lambda)p)$.

In other words, if $x < p$ and $y \in [x, p]$ then $x \prec y$. If $x > p$ and $y \in [p, x]$ then $x \prec y$.

Let R denote the set of all single peaked preferences over $[0, 1]$ and suppose that all players have single-peaked preferences. Can we design an incentive compatible mechanism?

2.2 Motivation for Single-peaked preferences

The definition above describes the scenario where each player i prefers a single value p_i and the farther one moves away from that p_i the more unhappy that player is. Examples are

- temperature in a shared office
- location of a store on a street
- income tax rates

Consider the following mehcanism. Apriori fix $x, y \in [0, 1]$. The mechanism takes bids (peak values) $p'_1, p'_2, ..., p'_n$ and chooses x or y depending on which of the two is prefered by the majority. It is easy to see that the mechanism is truthful. But this is kind of cheating since the mechanism has effectively reduced A to have size 2. This motivates the following definition.

Definition 2.2 (Onto rule) *A rule $f : R^n \to A$ is onto if there exists a preference relation $(\prec_1, ..., \prec_n)$ such that $f(\prec_1, ..., \prec_n) = x$ for all $x \in A$*

Is there an $f : R^n \to A$ such that

1. f is onto
2. f is incentive compatible

Clearly a dictatorship satisfies the above properties. We want to add a third property, namely anonymity, where f is invariant to permutations of players.

Definition 2.3 (Anonymous rule) *Given a preference relation* $(\prec_1, ..., \prec_n)$, *a rule* $f : R^n \to A$ *is anonymous if for all permutations* σ, $f(\prec_1, ..., \prec_n) = f(\sigma(\prec_1, ..., \prec_n))$

Is there an $f : R^n \to A$ such that

1. f is onto
2. f is incentive compatible
3. f is anonymous

A simple mechanism is to choose $f(p_1, ..., p_n) = \text{median}(p_1, ..., p_n)$

Exercise: Show that f satisfies the three desired properties.

We could also let $f(p_1, ..., p_n)$ be the kth ranked element of $p_1, ..., p_n$.
More generally, let $y_1, y_2, ..., y_{n-1}$ be arbitrary numbers in the interval $[0, 1]$

$$f(p_1, ..., p_n) = \text{median}(p_1, p_2, ..., p_n, y_1, ..., y_{n-1})$$

is truthful.

Remark 2.4 *One can show that this captures all the truthful functions that satisfy the three desired properties*

Theorem 2.5 *If* $f : R^n \to A$ *is onto, truthful, and anonymous then there exist* $y_1, y_2, ..., y_{n-1}$ *such that* $f(p_1, ..., p_n) = \text{median}(p_1, ..., p_n, y_1, ..., y_{n-1})$. *Note that we add* y_{n-1} *numbers and not* y_n *numbers in order for* f *to be onto.*

Proof: See book Theorem 10.2. □

Remark 2.6 *Note that other simple mechanisms such as average*$(p_1, ..., p_n)$ *etc. are not truthful.*

2.3 House exchange/ Kidney allocation problem

Suppose there are n players and each player i initially has a house labeled i and each player has strict preferences over the houses. We wish to allocate houses. Let A be the space of all allocations. Each $a \in A$ is essentially a matching from players to houses where player i is matched with house $a(i)$.

Let $\prec_i$ be preferences over A. If $\prec_i$ strictly orders over all $a \in A$ then the Gibbard-Satterthwaite theorem holds. Here we assume that i cares only about the house he is allocated and does not care about what others get. Thus, the space of preferences over A is restricted and is not the full set of permutations over A.

We use $\prec_i$ to indicate preferences of i on $\{1, 2, ..., n\}$ and not A.

Definition 2.7 *A matching $a \in A$ is said to be in the "core" if there is no $S \subset N$ such that S can break away and obtain a better utility. More formally, for $a \in A$, a set S is said to be blocking if there is another allocation $a' \in A$ such that*

1. *for all $i \in S$, $\quad a'(i) \in S$*
2. *for all $i \in S$, $\quad a'(i) \succ_i a(i)$ or $a'(s) = a(i)$ and there exists $j \in S$ such that $a'(j) \succ_j a(j)$*

Theorem 2.8 *Given any preferences $\prec_i$, the core is non-empty and there is precisely one matching in the core given by the top trading cycle (TTC) algorithm.*

The top trading cycle algorithm

1. Create a directed multigraph $G = (V, E)$
   a. V is the set of players
   b. for each i, if j is the kth ranked house on i's preferences $\prec_i$, add edge (i, j) and color it k. (Note loops are added).
2. for $k = 1$ to n do
   - Consider loops and cycles in current graph induced by edges of colors $\leq k$
   - **Observation** the set of cycles and loops is node disjoint
   - if $i_1 \to i_2 \to ... \to i_l \to i_1$ is a cycle, then assign i_2 to i_1, i_3 to i_2 ... i_1 to i_l
   - remove all verticies in cycles/loops found in this iteration

If preferences are strict, then the TTC algorithm will return a unique outcome.

Remark 2.9 *One can show that in any core, the assignment has to follow the assignment of the TTC algorithm and also the assignment found by the TTC algorithm is in the core.*

Remark 2.10 *Players have no incentive to misrepresent their preferences*

Theorem 2.11 *The TTC algorithm based mechanism is strategy proof*

Proof: Proof in book Theorem 10.7. □

The problem of housing allocation is similar to the problem of kidney exchanges. The situation is as follows. There is a list of patients waiting for a cadaver kidney. Suppose a patient on the bottom of the list finds a donor, but the donor is not compatible with the patient. However, it would be possible for the donor to give his kidney to someone at the top of the waiting list (who is compatible with the donor) in exchange for his spot on the list. An extention of the top trading cycle algorithm can be used in this case. (See Alvin Roth's paper on Kidney Exchange).

2.4 Stable Matchings

Consider a game with n men and n women. Each man has a (strict) ranking of the women and each woman has a (strict) ranking of the men. A matching is a pairing of men and women.

Definition 2.12 (Stable Matching) *A matching M is said to be stable if there are no pairs (m_1, w_1) and (m_2, w_2) in M such that m_1 and w_2 prefer to be with each other than with their current partners.*

The Gale-Shapley algorithm

1. While there is an some man who has a woman to propose to or each woman has at most one proposal
   a. Each man proposes to his top-ranked choice.
   b. Each woman who received ≥ 2 proposals (including the previously kept top choice) tentativley keeps her top ranked choice and rejects the rest.
   c. All men remove the women who rejected them from list.
2. Assign to each man the woman who has not rejected his proposal.

To see why the algorithm results in a matching notice that no man is assigned to more than one woman and since each woman only has one man on her list at any time, no woman is assigned to more than one man.

The Gale-Shapley algorithm shows the existence of a stable matching in all cases with strict preferences and is also a linear time algorithm. The algorithm is asymmetric. The male proposal version results in the following

- each man gets the best woman he can get in any stable matching
- each woman gets the worst man she can get in any stable matching
- men have no incentive to lie because of the first property
- women have an incentive to lie

In matching children to schools or residents to hospitals, one can use the male optimal algorithm with students/residents as men and schools/hospitals as women. School/hospitals can be trusted as they are more constrainted by laws and have disclosure rules.