CS 573: Algorithmic Game Theory	**Lecture date: Mar 5, 2008**
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1 LP Relaxation for social welfare and Walrasian equlibrium

1.1 LP Relaxation

Last class we wrote an LP relaxation for the winner determination (WD) problem

LPR

$$\begin{aligned} \max \quad & \textstyle\sum_{i,A} x(i,A)v_i(A) \\ & \textstyle\sum_{A\subseteq S} x(i,A) \le 1 \quad \forall \;\; bidder \;\; i \\ & \textstyle\sum_{i,A\ni j} x(i,A) \le 1 \quad \forall \;\; item \;\; j \\ & x(i,A) \ge 0 \end{aligned}$$

We write the dual of LPR as

DLPR

$$\begin{aligned} \min \quad & \textstyle\sum_{i\in N} u_i + \sum_{j\in S} p_j \\ & u_i + \textstyle\sum_{j\in A} P_j \ge v_i(A) \quad \forall \;\; A \subseteq S, \quad i \in N \\ & u_i, \quad p_j \ge 0 \end{aligned}$$

Interprelation of dual variables u_i - utility of i p_j - price of j

Lemma 1.1 *LPR and DLPR can be solved optimally with a polynomial (in n, m and max number of bits to represent any valuation) number of demand queries.*

One can show above using the ellipsoid method for linear programming.

From the above one can show that the VCG mechanism can be implemented in ploynomial time with demand queries if bidders are in "fractional" allocations.

Recall the definition of a Walrasian equilibrium from last class.

Definition 1.2 *Given prices $p_1, ..., p_n$ and allocation $S_1, ..., S_n$ is a Walrasian or Competition equilibrium if for each i, S_i is a demand for i at p and any unallocated item has price 0.*

We can show the following theorem.

Theorem 1.3 *Let $p_1^*, ..., p_m^*$ be a set of prices and $S_1^*, ..., S_n^*$ be an allocation at Walrasian equilibrium, then $S_1^*, ..., S_n^*$ is a welfare maximizing allocation. Moreover, the value of the allocation is at least as large as that of an optimum fractional allocation.*

Proof: Let x^* be an optimum fractional assignment - i.e. an optimum solution to LPR.
Since S_i^* is the demand of i at p^*

$$v_i(S_i^*) - p(S_i^*) \geq v_i(A) - p(A) \quad \forall A \subseteq S$$

$$\Rightarrow v_i(S_i^*) - p(S_i^*) \geq \sum_{A \subseteq S} x^*(i, A)(v_i(A) - p(A)) \tag{1}$$

Summing over all bidders we have

$$\sum_i (v_i(S_i^*) - p(S_i^*)) \geq \sum_i [\sum_{A \subseteq S} x^*(i, A)(v_i(A) - p(A))] \tag{2}$$

We wish to show that

$$\sum_i v_i(S_i^*) \geq \sum_i \sum_{A \subseteq S} x^*(i, A) v_i(A) \tag{3}$$

Therefore it suffices to prove that

$$\sum_i p(S_i^*) \geq \sum_i \sum_{A \subseteq S} x^*(i, A) p(A) \tag{4}$$

Since $S_1^*, ..., S_n^*$ is an allocation and an unallocated item has price 0, the LHS of 4 is the same as $\sum_j p_j^*$.
The RHS of 4 is

$$\begin{array}{rl} & \sum_i \sum_{A \subseteq S} x^*(i, A) p(A) \\ = & \sum_j p_j^* \sum_{i, A \ni j} x^*(i, A) \leq \sum_j p_j^* \end{array}$$

Since $\sum_{i, A \ni j} x^*(i, A) \leq 1$ by feasibility of allocation to LPR. □

We can also show

Theorem 1.4 *If LPR has an integral solution x^*, then the allocation $S_1^*, ..., S_n*$ defined by x^* is a Walrasian equilibrium with prices given by $p_1^*, ..., p_m^*$ which correspond to a dual solution to DLPR.*

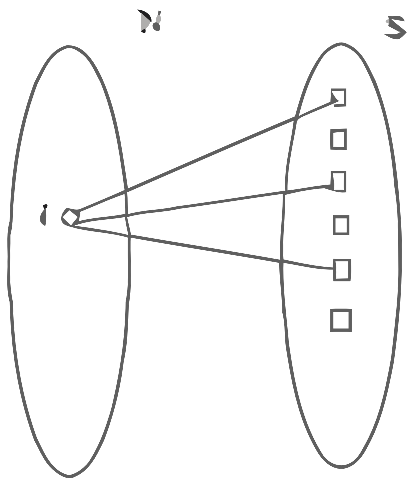

Figure 1: bipartite graph

Proof: Let x^* be an optimal solution to LPR and Let u^*, p^* be a dual optimal solution to x^*. Complementary Slackness Conditions on x^* and u^*, p^* imply the following.

$$\begin{aligned} & x^*(i, S_i^*) = 1 > 0 \\ \Rightarrow \quad & u_i^* = v_i(S_i^*) - p(S_i^*) \end{aligned}$$

By feasibility of u^*, p^* for DLPR for any $A \subseteq S$

$$\begin{aligned} & u_i \geq v_i(A) - p(A) \\ \Rightarrow \quad & v_i(S_i^*) - p(S_i^*) \geq v_i(A) - p(A) \\ \Rightarrow \quad & S_i^* \text{ is a demand for } i \text{ at } p^* \end{aligned}$$

Second suppose item j is unallocated $\Rightarrow \sum_{i,A \ni j} x(i, A) = 0 \ (< 1)$.
Thus constraint is not tight in u_i and for item j.
However if $p_j^* > 0$ constraint has to be tight, therefore $p_j^* = 0$. □

1.2 Matching Type Problem

An iteresting case when LPR has integral solution is for matching type problem.
More concretely,
S is a set of m items,
N is a set of n bidders.
Each bidder i wants only <u>one</u> item from S. For each $j \in S$, i values item j as $v_{ij} \geq 0$.
Note that $v_{ij} = 0 \Rightarrow$ i not really interested in j. So i is interested only in an item from $S_i = \{j \in S | v_{ij} \geq 0\}$
We can model this as a bipartite graph

How do we encode v_i as only interested in one item?
We set $v_i(A) = \max_{j \in A} v_{ij}$

Then LPR can be simplified to the following

$$\max \quad \sum_{i,j} v_{ij} x_{ij}$$
$$\sum_j x_{ij} \leq 1 \quad \forall \ i \in N$$
$$\sum_i x_{ij} \leq 1 \quad \forall \ j \in S$$
$$x_{ij} \geq 0$$
$$x_{ij} = 1 \text{ if } i \text{ gets } j \text{ , 0 otherwise.}$$

Dual of the above is

$$\min \quad \sum_i u_i + \sum_j p_j$$
$$u_i + p_j \geq v_{ij} \quad \forall \ i, j$$
$$u_i, p_j \geq 0$$

It is well known that matching polytope for bipartite graphs has integer solutions.
It follows from the theorem that prices given by p_j^* from the dual optimal solution and the allocation x_{ij}^* given by a primal solution from a Walrasian equilibrium.

Example 1
Consider the Vickery auction with only one item $S = \{1\}$. Item assigned to i with $max_i v_{i1}$.
What is the price? p_i = second highest price.
Only the highest bidder interested in item at this price.

Example 2
k identical items $S = \{1, 2, ..., k\}$, bidders interested in only one item, $v_i(S) = w_i$ if $|S| \geq 1$.
identical $\Rightarrow v_{i1} = v_{i2} = ... = v_{ik}$ $\forall i$.
What does VCG do?
Assign items to k highest bidders? Price? p_j = value of $k + 1$ highest bidder $\Rightarrow$ only the k highest bidders are interested at this price.

Example 3
m houses n bidders
each interested only in one house, each bidder i has seperate value v_{ij} for house j.
Theorem show that there are prices for houses such that each bidder who is allocated a house is happiest with that house at the given price.
Note that all bidders are simultaneously happy with the same prices.
Happiest in the sense of maximizing $v_{ij} - p_j$.

2 Randomized truthful mechanism for submodular/subadditive bidders

2.1 Submodular and Subadditive Valuations

Single-minded bidders care about only one set $A_i \subseteq S$. This is an extreme case of complements.
Now we consider the substitute-type valuations.

Definition 2.1 *A function $f : 2^S \to \mathbb{R}$ is submodular if and only if*

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \quad \forall \ A, B \subseteq S.$$

Alternatively f is submodular if and only if

$$f(A + e) - f(A) \geq f(B + e) - f(B) \quad \forall \ A \subseteq B, \ \forall \ e \in S.$$

Definition 2.2 *A function $f : 2^S \to \mathbb{R}$ is sub-additive if and only if $\forall \ A, B \subseteq S$*

$$f(A) + f(B) \geq f(A \cup B).$$

2.2 Submodular/Subadditive Bidders

Suppose all bidders are submodular(or subadditive). Can we get a good algorithm for maximizing social welfare via a truthful mechanism? We don't know a deterministic mechanism, instead we give a randomized mechanism.

Definition 2.3 *A mechanism that uses randomness is truthful if it is a random choice over deterministic truthful mechanisms.*

Definition 2.4 *The social welfare of a randomized mechanism is the expected value of the social welfare.*

Note that when each bidder i's valuation v_i is submodular(subadditive), we still have a communication complexity issue.
So we want to have a polynomial time mechanism.

First we consider a simple case.
Assume mechanism <u>knowns</u> $H = \max_i V_i(S)$.

2.3 Randomized Mechanism

- Pick a random price $p \in [H/m, H]$.
- Order bidders arbitrarily.
- $S' = S$.
- for $i = 1$ to n do
    $S_i \leftarrow$ best set demanded by i from S' with each item at price p
    $S' \leftarrow S' \backslash S_i$.

Theorem 2.5 *Mechanism is truthful and expected social welfare is $\Omega(\frac{OPT}{\log m})$, where OPT = optimal social welfare, $m = |S|$. (for both Submodular and Subadditive).*

However we may not know H. To find out H we can alter the mechanism which makes it some what more complex.

2.4 Randomized Mechanism 2

- Partition bidders into two random groups STAT and FIXED. Each bidder i is put in STAT with probablity $\frac{1}{2}$ and FIXED with probability $\frac{1}{2}$ (independently).
- Run a second-price auction on bidder in STAT only by bundling all of S into one item. Let H be the max price offered.
- With probability $\frac{1}{2}$ assign all of S to winner in above step.
- With probability $\frac{1}{2}$ continue.
- Run previous mechanism with H and bidders in FIXED only.

Theorem 2.6 *Mechanism is truthful and expected social welfare is* $\Omega(\frac{OPT}{\log m})$.