CS 573: Algorithmic Game Theory	**Lecture date: 29th February, 2008**
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In the last lecture, we looked at the VCG mechanism and realized that the Winner determination(WD) problem is NP-Hard even in a case as simple as that of the single minded bidders. Continuing on single minded bidder case, we also saw that an $m^{1/2-\epsilon}$ approximation for WD problem is also hard to compute unless $P-NP$. We developed a $\sqrt{m}$ approximation algorithm: which orders the bidders on the basis of the quantity :$\frac{b_i}{\sqrt{|B_i|}}$ and then picks in this order whenever possible.

However, we realized that we cannot directly use an approximation algorithm in VCG mechanism as it can lead to "insensible" results, like asking a bidder to pay more than his bid. In the first section, we shall try to develop a payment scheme which generates sensible payments. We shall also try to generalize to find out the properties of a mechanism which which ensure truthful behaviour. In the second section, we shall have a look at the communication complexity of the VCG auctions.

1 Mechanism Design

In this section, we assume that (A_i, w_i) represents the actual set wanted by player i and the true value of that set, respectively and (B_i, b_i) the actual bid submitted by player i. Consider a payment scheme for the $\sqrt{m}$ algorithm we had developed in the last lecture. Recall that we have n players and m items to be auctioned and the algorithm reorders the players such that

$$\frac{b_1}{|\sqrt{B_1}|} \geq \frac{b_2}{|\sqrt{B_2}|} \geq \ldots \geq \frac{b_n}{|\sqrt{B_m}|}$$

and then allocates in this order. Consider the following payment scheme for this setting:

- If i is a loser then $p_i = 0$.
- If i is a winner then $p_i = \frac{b_j|\sqrt{B_i}|}{|\sqrt{B_i}|}$ where $j = min\{t : B_i \cap B_t \neq \phi, \forall k < t$ such that k is winner and $k \neq i, B_k \cap B_t = \phi\}$. Intuitively, p_i is the amount such that if b_i was any lesser than this, then i would lose. This can be seen from the way p_i is computed, because if b_i was less than the suggested p_i, then player j, who is currently behind i in the ordering, will win and at least one of the elements in B_j is also present in B_i which will preclude i from winning.

Clearly this payment scheme generates p_i's which is less than or equal to b_i. Moreover, we also claim that this scheme will ensure truthful behaviour. But for that, we first need to analyze certain properties.

1.1 Ensuring truthful behaviour

Consider the following two properties of a mechanism:

1. **Monotonicity:** If a player i wins with the bid:(B_i, b_i) then for any $B'_i \subseteq B_i$ and $b'_i \geq b_i$, he must win with the bid (B'_i, b'_i). In simple words, if the player wins in a certain case, then he must also win if he reduces the set of items he desires and is willing to pay at least as much. Naturally, this property concerns the algorithm chosen for picking the winners.

2. **Criticality:** The payment scheme satisfies criticality if for a winner i with a bid (B_i, b_i), the payment p_i is the smallest x such that (B_i, x) is still a winner. This property concerns the payment scheme of the mechanism.

Now given these properties, we state the following important result

Theorem 1.1 *A mechanism which makes the losers pay 0 amount is truthful iff it satisfies monotonicity and criticality.*

Proof: Consider a mechanism with the aforementioned properties. We shall prove only the *if* part. Consider a player i and fix the bids for all the other players. We wish to show that (A_i, w_i) is the utility maximizing bid.

Firstly, we make the following simple observation: Truthful bidders have non-negative utilities. This can be easily seen from the criticality property, which ensures that $\forall i, p_i \leq b_i$. And hence if $b_i = w_i$ then $w_i - p_i \geq 0$.

Secondly, we can assume that $B_i \supseteq A_i$ because if this is not the case then even if i wins, the value of B_i is zero leading to a non-positive utility (which is no better than the always non-negative utility of truthful bidding).

Thirdly, if (B_i, b_i) is a losing bid then it has zero utility which is no better than the utility of (A_i, w_i)(which we have shown to have non-negative utility). Hence if i is to lose, then he might as well bid truthfully which won't give a worse utility. So from now on, we consider that (B_i, b_i) is a winning bid. Now we prove the following claim:

Claim 1.2 *The utility of (A_i, b_i) is at least as large as the utility of (B_i, b_i).*

Proof: Since $A_i \subseteq B_i$ and (B_i, b_i) is a winning bid, so by monotonicity (A_i, b_i) is also a winning bid.

Now, let p_i be the payment for the bid (B_i, b_i). By criticality, (B_i, p_i) is a winning bid and by monotonicity (A_i, p_i) is a winning bid too. Let p'_i be the payment for (A_i, b_i). We wish to prove that $p'_i \leq p_i$. Suppose $p'_i > p_i$, then by the criticality property we have that for all $x < p'_i$, in particular for some $x \in (p_i, p'_i)$, (A_i, x) is a losing bid. But this clearly violates the monotonicity property since we know that (A_i, p_i) is a winning bid. Hence we have that $p'_i \leq p_i$ which along with the fact that (A_i, b_i) is a winning bid implies that the utility of the bid (A_i, b_i) is no worse than the utility of the bid (B_i, b_i). $\square$

Now we make another claim which along with the previous claim would imply the result

Claim 1.3 *The utility of (A_i, w_i) is at least as large as the utility of (B_i, b_i).*

Proof: We consider different possible scenarios in this case:

1. $\underline{b_i < w_i}$: Let p_i be the payment with bid b_i. By monotonicity, (A_i, w_i) is a winning bid and since the payment is independent of the amount which is bid(by criticality), so the payment in this case is also p_i. Hence the utility of bidding (A_i, w_i) is the same as the utility of bidding (A_i, b_i).

2. $\underline{b_i > w_i}$: Let p_i be the payment with the bid (A_i, b_i). We have two more cases in these scenario:

   - $\underline{p_i \leq w_i < b_i}$: By criticality, (A_i, w_i) is still a winning bid and since p_i is independent of the amount bid, so the utility of bidding (A_i, b_i) is the same as that of bidding (A_i, w_i).
   - $\underline{w_i < p_i \leq b_i}$: Utility of $(A_i, b_i) = w_i - p_i \leq 0$ and we know that the truthful bid has non-negative utility. Hence the utility of (A_i, w_i) is no worse than that of (A_i, b_i).

So clearly, we have established that the utility of bidding (A_i, w_i) is no worse than that of bidding (A_i, b_i). □

It is easy to see that both the claims we have proved, establish that the utility of bidding (A_i, w_i) as at least as larger as that of bidding any other (B_i, b_i). □

Clearly, the approximation algorithm we have suggested satisfies monotonicity property. Also, the suggested payment scheme satisfies the criticality property due to the very way it defines the payments. Hence the $\sqrt{m}$ approximation algorithm along with the suggested payment scheme for the single-minded bidders case, ensures truthful behaviour. So we get the following theorem

Theorem 1.4 *For the single minded bidders case, there exists a polynomial time mechanism which gives a $O(\sqrt{m})$ approximation to the social welfare and ensures truthful behaviour.*

2 Communication Complexity

In the previous sections, we assumed the single-minded bidder case which requires very little communication($O(mn)$ that is). Now in this section we look at the issue of communication complexity.

To ensure that the issues of communication and computation are separately dealt with, we assume that the auctioneer is computationally unbounded. Also to eliminate precision related issues, assume that the bidders have 0/1 valuations that is $v_i(A) = 0$ or 1, $\forall A \subseteq S$. Moreover we assume that the valuations are monotone that is $V_i(A) \geq V_i(B)$ if $A \supseteq B$. Recall that the social welfare is the sum of the respective values of the items obtained by the winners. We state the following results without proof.

Theorem 2.1 *Even for two players with 0/1, monotone valuations, any mechanism that maximizes social welfare requires $\binom{m}{m/2}$ bits of communication*

In fact, we can make a stronger statement regarding the communication complexity involved in even approximating social welfare.

Theorem 2.2 *Any mechanism which achieves an approximation strictly smaller than* $\min\{n, m^{1/2-\epsilon}\}$ *for social welfare requires exponential amount of communication.*

The profoundness the last result can be understood from the fact that an n-approximation to the social welfare can be trivially achieved for monotone valuations with $O(n)$ communication as follows: Combine all the items in one set and auction them to the highest bidder. Since there are a total of n bids, each for a non-empty set of items, so this is clearly an n-approximation to the social welfare. Obtaining a $\sqrt{m}$ approximation with reasonable amount of communication is slightly harder, but can be achieved.

2.1 Issues in communicating

There are 2 approaches towards communicating with the bidders.

1. BIDDING LANGUAGES: The bidders use simple bidding language to indicate the set(s) they are interested in and the amount they are willing to pay for it. For the single minded case, the bidders just communicate the set they are interested in. These languages allow scope for logical operators like OR, XOR, etc. For example if a bidder is interested in two different sets, each with their own valuations, then he can use the OR operator to specify the sets he is interested in.

2. QUERY BIDDERS: In this case, the auctioneer queries the bidders regarding their valuations etc. Ideally the number of queries should be polynomial in m and n. In general, the literature mentions two types of queries most prominently:

   (a) **Value Queries:** The auctioneer can ask queries to a player i like: What is $V_i(A)$ for some $A \subseteq S$.?

   (b) **Demand Queries:** The auctioneer can ask queries like: given prices $p_1, p_2, \ldots, p_m$ for the items, which set $A \subseteq S$ maximizes $V_i(X) - p(X)$ over all $X \subseteq S$, where $p(X) = \sum_{j \in X} p_j$?

   Clearly demand queries are at least as powerful as value queries(any value query can be posed as a demand query by setting all the prices zero). In fact they are much powerful than value queries in the following sense:

   - One can show that for even simple compact valuation functions, demand queries can be NP-Hard to answer.
   - One can also show that in some cases, answering a demand query may require an exponential number of value queries.

   These hardness results are for arbitrary valuations. The inherent difference between these two kind of query models is the place where computation is carried out. In the value query case, the computation usually occurs at the auctioneer's side while in the demand query case, the computation happens at the bidders' side. Philosophically one can argue in favour of demand query model by saying that the bidders should have simple enough valuations so that they can solve the demand queries easily. This matter, however, is very much arguable.

3 Fairness and VCG Mechanism

Lets first have a look at an example which demonstrates a certain kind of unfairness involved in the VCG mechanism.

Consider the auction of two items: $\{a, b\}$. There are two bidders: $\{1, 2\}$ with $V_1(\{a\}) = V_1(\{b\}) = 12, V_1(\{a, b\}) = 13, V_2(\{a\}) = V_2(\{b\}) = 12$, and $V_2(\{a, b\}) = 20$. The VCG mechanism, in order to maximize social welfare, allocates $S_1 = \{a\}$ and $S_2 = \{b\}$. Now according the Clark's pivot rule the payments charged are: $p_1 = 20 - 12 = 8$ and $p_2 = 13 - 12 = 1$. Here, player 1 has to pay more than player 2, even though both of them get one item each which both of them value equally. Clearly, VCG mechanism is not being completely fair.

Now we define certain conditions which in some sense represent ideal conditions.

Definition 3.1 *Given prices $p_1, p_2, \ldots, p_m$ for items, a set S_i is said to be demanded by bidder i if $V_i(S_i) - p(S_i) \geq V_i(A) - p(A),\ \forall A \subseteq S$.*

Definition 3.2 *A set of prices $p^* = p_1^*, p_2^*, \ldots, p_m^*$ and an allocation $S_1^*, S_2^*, \ldots, S_m^*$ is a Walrasian competitive equilibrium if $\forall i, S_i^*$ is demanded by i given p^* and $\forall$ items j such that j is not allocated, $p_j^* = 0$.*

Roughly speaking, Walrasian Equilibrium defines an ideal condition in which everyone gets exactly what he or she wants, given the prices and there is contention for items between any two players.

Now lets have a look at it in the context of the Winner Determination Problem. Lets express the WD problem in the form a giant ILP problem. Let $x(i, A)$ be a boolean variable which indicates if A is allocated to player i or not where $A \subseteq S$. So consider the following ILP

$$\begin{aligned} \text{maximize} \quad & \sum_{i, A \subseteq S} v_i(A) x(i, A) \\ \text{subject to} \quad & \sum_{A \subseteq S} x(i, A) \leq 1 \quad \forall i \\ & \sum_{i=1}^{n} \sum_{A : j \in A} x(i, A) \leq 1 \quad \forall j \in S \\ & x(i, A) \in \{0, 1\} \end{aligned}$$

The above ILP with exponential number of constraints and variables, clearly models the WD problem. We relax this ILP into an LP (and call it LPR), by relaxing the constraint $x(i, A) \in \{0, 1\}$ to $x(i, A) \geq 0$.

Given the LPR, we have the following results, the details which we shall delve into in the next lecture:

Theorem 3.3 *Let $p_1^*, p_2^*, \ldots, p_m^*$ and $S_1^*, S_2^*, \ldots, S_m^*$ be a Walrasian Equilibrium then this allocation is actually a welfare maximizing allocation, given the prices. Moreover, the value of this allocation is at least as large as the value of fractional allocation attained by LPR.*

Theorem 3.4 *If LPR has an integral optimum, then there exists a Walrasian Equilibrium*

Walrasian equilibrium is an idealized condition but there are several real life examples in which it can be achieved by careful pricing (for example in housing allocations). We shall look at it in a detailed way in the next class.