CS 573: Algorithmic Game Theory	**Lecture date: Feb 27, 2008**
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1 Review: VCG Mechanism for Combinatorial Auction

1.1 Notations

- m - number of items
- S - set of items
- N - set of bidders
- n - number of bidders ($=|N|$)
- $v_i : 2^S \to R$ - valuation of bidder i for all possible subsets

1.2 Mechanism

1. Receive bids, $b_i : 2^S \to R$,
2. Compute allocation $(S_1^*, \cdots, S_n^*)$ that maximizes $\sum_j b_j(S_j^*)$,
3. Charge payment $p_i(b) = h_i(b_{-i}) - \sum_{j \neq i} b_j(S_j^*)$ where $h_i(b_{-i}) = \max_{\{S_j\}_{j \neq i}} \sum_{j \neq i} b_j(S_j)$

Implementing the VCG mechanism requires

1. (Communicative complexity) For each bidder i to communicate bid b_i to the auctioneer, and
2. (Computational complexity) The mechanism needs to compute an optimum allocation based on the bids

Unfortunately to communicate a total bid requires 2^m numbers. We are interested in mechanisms that have communicative and computational requirements that are polynomial in n and m.

2 Computational Issue in VCG Mechanism

2.1 Single-Minded Bidder

Definition 2.1 *A bidder i is single-minded if $\exists A_i \subseteq S$ and $w_i \in R$ such that*

$$v_i(X) = \begin{cases} w_i & \text{if } A_i \subseteq X, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

That is, bidder i is only interested in getting all of S_i or more.

Every bidder i is single-minded and this fact is common knowledge. To implement VCG mechanisms the bidders now only have to send two pieces of information (B_i, b_i) where $B_i \subseteq S$ and $b_i \in R$. Note that an untruthful bidder can lie and have $B_i \neq A_i$ or $b_i \neq w_i$. In the single-minded bidder case the required communication is polynomial in n and m. Thus we can focus on the second aspect, namely the computation of the optimum allocation.

2.1.1 Winner Determination Problem

Recall that the VCG mechanism needs to find an allocation $(S_1, S_2, \cdots, S_n)$ that maximizes $\sum_j b_j(S_i)$. In the single-minded case, $S_i = B_i$ or $S_i = 0$. Bidder i is a winner if $S_i = B_i$ and a loser if $S_i = 0$. Thus the allocation problem in the single-minded case is to determine the winner to maximize their value. In conventional auctions this is called the winner determination (WD) problem. Unfortunately we can show that the WD problem is NP-hard.

Definition 2.2 *Given undirected graph $G = (V, E)$, MIS is to find a maximum independent set $X \subseteq V$ in G. A subset of vertices $Y \subseteq V$ is independent if $\forall u, v \in Y, (u, v) \notin E$.*

Theorem 2.3 *Winner determination (WD) problem is NP-hard.*

Proof: Given $G = (V, E)$ we consider n bidders where bidder i corresponds to $v_i \in V$, $n = |V|$. Let $S = E$. Define $B_i = \{e | e \text{ is incident to } v_i\}$ and $b_i = 1$. It is easy to verify that $Y \subseteq V$ is an independent set if and only if the allocation $(S_1, \cdots, S_n)$ with $S_i = B_i$ if $v_i \in Y$ and $S_i = 0$ if $v_i \notin Y$ is feasible (i.e. $S_i \cap S_j = \emptyset, i \neq j$). Since $b_i = 1, \forall i$ maximizing welfare is same as finding a maximum independent set in G. Since MIS is NP-hard, WD is also NP-hard. □

We observe that MIS is NP-hard even in bounded degree graphs. Thus WD is NP-hard even in cases where $|B_i|$ is bounded for all i.

Given that the WD problem is NP-hard, what can we do? Two natural questions:

1. Is there a polynomial time approximation algorithm that can be used to find an approximation to social welfare?

2. Can we use it in some alternative mechanism (since VCG mechanism requires optimal solutions to the WD problem) that is truthful?

2.1.2 Approximation for Winner Determination Problem

Definition 2.4 *An α-approximation for the WD problem is a polynomial time algorithm that on all instances $\mathcal{I}$ of WD returns a solution of value at least $\frac{OPT(\mathcal{I})}{\alpha}$.*

Note that α can be a function of input size $|\mathcal{I}|$.

Unfortunately, we can show the following:

Theorem 2.5 *Unless P=NP, there is no $m^{1/2-\varepsilon}$ approximation to WD for any $\varepsilon > 0$.*

Proof: It is known by a famous result of Haståd based on fundamental work in TCS that unless P=NP, there is no $n^{1-\varepsilon}$ approximation for the MIS problem where n is the number of vertices in the input graph. One can see from the reduction from MIS to WD that if $\mathcal{I}$ is the instance produced from G, then $OPT(\mathcal{I}) = OPT(G)$ and moreover, if A is any solution to $\mathcal{I}$ of value x then there is a solution of value x in G. Note, however, that m, the number of items in $\mathcal{I} \leq \binom{n}{2}$. Therefore an $n^{1-\varepsilon}$ hardness for MIS implies and $m^{1/2-\varepsilon}$ hardness for WD. □

We will now develop an $\mathcal{O}(\sqrt{m})$ approximation to WD. We can imagine three natural greedy algorithms as follows.

Definition 2.6 (Greedy 1) *Order bids in non-increasing b_i values (i.e., $b_1 \geq b_2 \geq \cdots \geq b_n$). Set $S_i = B_i$ if $S_i \cap S_j = \emptyset, j < i$, otherwise $S_i = 0$.*

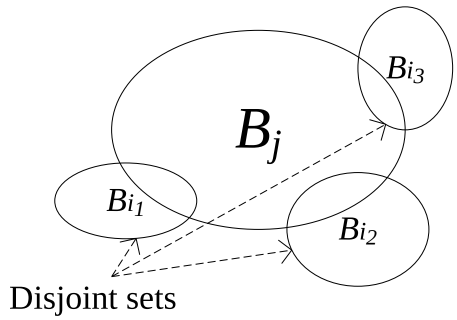

Figure 1: Bidder j blocks bidder i_1, i_2 and i_3. Since the blocked bidders are winners in OPT allocation, they have disjoint sets of B_i.

Suppose $B_1 = S_1$, $b_1 = 1+\varepsilon$ and $B_i = \{i-1\}, b_i = 1$ for $i = 2, \cdots, m+1$. Although all bidders except for bidder 1 could have sum value of m (i.e., $OPT = m$), bidder 1 takes all items and its value is only $1+\varepsilon$. Thus Greedy 1 is not a good approach.

Definition 2.7 (Greedy 2) *Order bidders accordingly to non-increasing* $\frac{b_i}{|B_i|}$ *values (i.e.,* $\frac{b_1}{|B_1|} \geq \frac{b_2}{|B_2|} \geq \cdots \geq \frac{b_n}{|B_n|}$*). Greedily allocate as before.*

Suppose $b_1 = 1+\varepsilon, B_1 = \{1\}$, $b_2 = m$ and $B_2 = S$. Bidder 1 takes item 1, having value $1+\varepsilon$ by Greedy 2. In this case bidder 2 could have had value of m, which is OPT.

Exercise 1 *Show that both Greedy 1 and Greedy 2 are* $\Omega(m)$*-approximates (clearly they are* m*-approximations. Why?).*

Exercise 2 *Greedy 1 and 2 give a* d*-approximation if* $|B_i| \leq d, \forall i$.

Definition 2.8 (Greedy 3) *Order bidders accordingly to non-increasing* $\frac{b_i}{\sqrt{|B_i|}}$ *values (i.e.,* $\frac{b_1}{\sqrt{|B_1|}} \geq \frac{b_2}{\sqrt{|B_2|}} \geq \cdots \geq \frac{b_n}{\sqrt{|B_n|}}$*). Greedily allocate as before.*

We can prove the following:

Lemma 2.9 *Greedy 3 is a* $\sqrt{m}$*-approximation for WD.*

Proof: Let X be a set of winners in an optimal allocation. Then $\sum_{i\in X} b_i = OPT$. Let Y be the winners in Greedy 3. We want to show that $\sum_{i\in Y} b_i \geq \frac{OPT}{\sqrt{m}}$.

Let $Z = X \setminus Y$ be winners in X who were not picked in Y. Let $Y' = Y \setminus X$ be winners in Y who were not in X. We will show that $\sum_{i\in Z} b_i \leq \sqrt{m} \sum_{i\in Y'} b_i$, which implies that

$$\begin{aligned}\sum_{i\in X} b_i &= \sum_{i\in Z} b_i + \sum_{i\in X\cap Y} b_i \\ &\leq \sqrt{m}\sum_{i\in Y'} b_i + \sum_{i\in X\cap Y} b_i \\ &\leq \sqrt{m}\sum_{i\in Y} b_i \qquad (2)\end{aligned}$$

Consider $i \in Z$. Since Greedy did not pick i when considering i, we can say that bidder i was blocked by $j \in Y$ (i.e., $B_i \cap B_j \neq \emptyset$) and moreover, $j \in Y'$ (why? if i were blocked by $j \in Y \setminus Y'$, then i could not have been in Z.). Note that there could be multiple j that block i.

For each $i \in Z$, we assign a unique $j \in Y'$ that blocks it by picking the smallest j amongst all that block i. Let $Z_j = \{i \in Z | i \text{ is blocked by } j \text{ and } j \text{ is smallest blocker}\}$. Note that $Z_j \cap Z_{j'} = \emptyset, j \neq j'$.

Claim: $\sum_{i \in Z_j} b_i \leq \sqrt{m} b_j$

Note that for $i, i' \in Z_j$, $B_i \cap B_{i'} \neq \emptyset$ since $i, i' \in X$. Since Greedy considered j before any $i \in Z_j$, we have

$$\frac{b_i}{\sqrt{|B_i|}} \leq \frac{b_j}{\sqrt{|B_j|}}$$

$$b_i \leq \frac{b_j}{\sqrt{|B_j|}} \sqrt{|B_i|}$$

$$\sum_{i \in Z_j} b_i \leq \frac{b_j}{\sqrt{|B_j|}} \sum_{i \in Z_j} \sqrt{|B_i|}, \tag{3}$$

where we can apply Cauchy-Schwartz inequality $\sum_{i \in Z_j} \sqrt{|B_i|} \leq \sqrt{|Z_j|} \sqrt{\sum_{i \in Z_j} |B_i|}$, resulting in

$$\sum_{i \in Z_j} b_i \leq \frac{b_j}{\sqrt{|B_j|}} \sqrt{|Z_j|} \sqrt{\sum_{i \in Z_j} |B_i|}. \tag{4}$$

We know that $|Z_j| \leq |B_j|$ since $i, i' \in Z_j$ and $B_i \cap B_{i'} = \emptyset$. In addition, we know $\sqrt{\sum_{i \in Z_j} |B_i|} \leq \sqrt{m}$ since $Z_j \subseteq X$. By using these, we finally have

$$\sum_{i \in Z_j} b_i \leq \sqrt{m} b_j, \tag{5}$$

which completes the proof. □

Example: Let $B_1 = S_1$ and $b_1 = \sqrt{m} + \varepsilon$. For the rest, let $B_i = \{i - 1\}$ and $b_i = 1$ for $i = 2, 3, \cdots, m + 1$. Assume players bid truthfully, i.e., $B_i = S_i$ and $b_i = v_i$. Greedy 3 will assign S to player 1 and no items are left, having value of the allocation $\sqrt{m} + \varepsilon$. If player 1 is removed then Greedy 3 will assign B_i to i for $i = 2, \cdots, m + 1$. This gives the value of m. If this approximate algorithm is directly used in VCG mechanism, $p_1 = m - (\sqrt{m} + \varepsilon)$, which is larger than b_1! More generally, approximate algorithms for WD problem may result in payment which is larger than a bid of the winner. This is because an approximation algorithm that cannot guarantee monotonicity in removing a player may increase the value of the allocation complied by the algorithm. In the next lecture we will discuss on mechanisms that guarantee the monotonicity and thus are truthful.

Exercise 3 (Greedy 4) *Prove that the following approximate algorithm gives $2\sqrt{m}$-approximation. Greedy 4 either picks the maximum profit bid or the bid which Greedy 2 picks up. That is, it picks whichever gives more profit.*