CS 573: Algorithmic Game Theory	**Lecture date: 22 February 2008**
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Overview

Vickrey auctions focus on allocating a single item in a way that maximizes welfare. In this section, we study the allocation of multiple items that achieves a social optimum. A natural extension of the signle-item case is to auction each item independently. However, bidders may value *bundles* of items more than the sum of their parts – like the ingredients for a cake – or less – like a jacket and a coat –, thereby making single-item auctions impractical.

To allow the players to express synergies between different items, we introduce the concept of *combinatorial auctions*. In a combinatorial auction, players are allowed to express these synergies by bidding for *bundles* or *combinations* of items instead of individual items. As in single-item auctions, the players may behave strategically by entering bids that differ from what their values for the bundles actually are. Therefore, we introduce the Vickery-Clarke-Groves or VCG mechanism which is an extension of Vickery auctions to combinatorial auctions in that it forces the players to reveal their true valuations.

Deregulation in various industries has spurred research into combinatorial auctions. Indeed, they are used to sell various public assets to the players that value them the most, in other words to the players who can use them most efficiently, thereby maximizing overall welfare. Of note are airport take-off and landing slot and wireless spectrum auctions.

1 Combinatorial Auctions

Notation 1.1 *Bidders, items, valuation functions*
We denote by $N \simeq [\![1, n]\!]$ *the set of bidders, by* $S \simeq [\![1, m]\!]$ *the set of items being auctioned and we assume that each bidder* i *has a valuation function* $v_i : \mathcal{P}(S) \to \mathbb{R}$ *where* $\mathcal{P}(S)$ *is the power set of* S *.*

We interpret v_i as the value of $A \subseteq S$ according to player i. A valuation need not be increasing or even non-negative. We will consider special classes of valuation functions in due course.

Example 1.2 *Additive valuations*
We say that a valuation $v : \mathcal{P}(S) \to \mathbb{R}$ is additive *iff there is a weight function $\omega : S \to \mathbb{R}$ such that*

$$\forall A \in \mathcal{P}(S) \quad v(A) = \sum_{a \in A} \omega(a)$$

If a player i has an additive valuation it means that he values any item irrespective of what other items he gets. Further, if all players have additive valuations, we may auction each item separately instead of performing a combinatorial auction.

However, valuations aren't generally additive and they capture more complicated relationships between the different items' usefulness. In particular they capture two broad aspects:

Definition 1.3 *Substitutes, complements*
We say that two items are substitutes if the value of having both is less than the sum of their values. We say they are complements if the value of having both is more than the sum of their values.

For example, from a consumer's perspective, bread and rice are substitutes. Indeed, getting both is less valuable than the sum of their values separately. Likewise, the left and right shoes of a pair of shoes are complements.

Definition 1.4 *Quasi-linear utilities*
We say that the utility for player i is quasi-linear iff $u_i = v_i(S_i) - p_i$ where S_i denotes the bundle i receives as a result of the auction and p_i the price it is charged for it.

Definition 1.5 *Allocation*
We say that $(P_1, \ldots, P_n) \in \mathcal{P}(S)^n$ is an allocation iff it is a partition of a subset of S, i.e.

- $\bigcup_{i=1}^{n} P_i \subseteq S$
- $\forall (i, j) \in N^2 \quad i \neq j \Rightarrow S_i \cap S_j = \emptyset$

Definition 1.6 *Socially optimal/efficient/welfare-maximizing allocation*
We say that an allocation $(P_1, \ldots, P_n)$ is socially optimal/efficient/welfare-maximizing iff

$$\forall (Q_1, \ldots, Q_n) \text{ allocation} \quad \sum_{i=1}^{n} v_i(Q_i) \leq \sum_{i=1}^{n} v_i(P_i)$$

In other words, a socially optimal allocation maximizes the total value bidder's receive from the allocation. Our goal is therefore to find a socially optimal allocation in a setting where bidders have quasi-linear utilities they are trying to maximize.

2 The Vickrey-Clarke-Groves (VCG) Mechanism

The Vickrey-Clarke-Groves mechanism is an extension of the Vickrey single-item auction to combinatorial auctions. It is a truthful mechanism that is efficient in that it maximizes social welfare. We will ignore computation and communication issues and focus on the mechanism's correctness.

Like in the previous section, we denote by n and m the number of players N and items S respectively. We also assume that each player has a private valuation function $v_i : \mathcal{P}(S) \to \mathbb{R}$ and that there is no collusion.

Notation 2.1 *Bid functions, allocation, payments*
Each player i submits a bid function $b_i : \mathcal{P}(S) \to \mathbb{R}$ *to the mechanism (auctioneer). The mechanism then decides an* allocation $(S_1, \ldots, S_n) \in \mathcal{P}(S)^n$ *and (possibly negative)* payments $(p_1, \ldots, p_n) \in \mathbb{R}^n$.

Definition 2.2 *Vickrey-Clarke-Groves Mechanism (VCG)*
The Vickrey-Clarke-Groves mechanism works as follows on receiving bids b_i

1. *Compute an allocation $(S_1^*, \ldots, S_n^*)$ that maximizes $\sum_{i=1}^n b_i(S_i^*)$. In other words*

$$(S_1^*, \ldots, S_n^*) = \underset{(S_i)\ \textit{allocation}}{\operatorname{argmax}} \sum_{i=1}^n b_i(S_i)$$

2. *Compute prices $(p_1, \ldots, p_n)$ appropriately (see below)*

3. *Allocate S_i^* to each player i and charge them p_i*

Proposition 2.3 *Optimality of VCG under truthfuness*
If the players are truthful (i.e. $\forall i \in N\, b_i = v_i$) then the VCG mechanism maximizes social welfare (it is efficient).

We choose the payments so that the bidders bid truthfully:

Notation 2.4 *Bid vector, price functions*

- $\bar{b} =_{\text{def}} (b_1, \ldots, b_n)$ *is the vector of bids.*
- $\forall i \in N \quad b_{-i} =_{\text{def}} (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ *is the vector of bids other than i's.*
- $\forall i \in N \quad p_i(\bar{b}) \in \mathbb{R}$ *gives the price for i given bids $\bar{b}$ (formally, $p_i : (\mathbb{R}^{\mathcal{P}(S)})^n \to \mathbb{R}$).*

Definition 2.5 *Generic VCG mechanism*
Let $(h_i)_{i \in N}$ be n $\mathbb{R}^n \to \mathbb{R}$ functions such that h_i is independent of the i-th coordinate. That is $h(\bar{b}) = h(b_{-i})$ (by abuse of language) or, in other words:

$$\forall i \in N \quad h(b_1, \ldots, b_n) = h(b_1, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_n)$$

We call generic VCG mechanism *the VCG mechanism described in definition 2.2 that uses the price functions*

$$p_i(\bar{b}) = h_i(b_{-i}) - \sum_{j \neq i} b_j(S_j^*)$$

Theorem 2.6 *Truthfulness of generic VCG mechanisms*
Every *generic VGC mechanism is truthful (i.e. incentive-compatible or strategy-proof).*

Proof: Fix i, b_{-i} and $(h_i)_{i\in N}$. If i declares v_i, the mechanism computes an allocation $(S_1, \ldots, S_n)$ and $i's$ utility is $u_i(v_i, b_{-i}) = v_i(S_i) + \sum_{j\neq i} b_j(S_j) - h_i(b_{-i})$
If he enters any other bid b_i, the algorithm computes a possibly different allocation $(S_1' \ldots S_n')$ and his utility is $u_i(b_i, b_{-i}) = v_i(S_i') + \sum_{j\neq i} b_j(S_j') - h_i(b_{-i})$
Since by definition 2.2 $(S_1, \ldots, S_n) = \text{argmax}_{(A_i)\text{ alloc.}}\{v_i(A_i) + \sum_{j\neq i} b_j(A_j)\}$, we have, particular, $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$. This means that choosing $b_i = v_i$ is a utility-maximizing strategy. $\square$

An interpretation of VCG is:

$$u_i(\bar{b}) = \underbrace{v_i(S_i^*) + \sum_{j\neq i} b_j(S_j^*)}_{=\sum_{j=1}^n v_j(S_j^*)\text{ assuming } b_j=v_j} - \underbrace{h_i(b_{-i})}_{\text{does not depend on i}}$$

Therefore, each bidder has an interest in maximizing social welfare $\sum_{j=1}^n v_j(S_j^*)$.

Now, the h_i functions may be arbitrary. In particular, they may each be identically zero over $\mathbb{R}^n$, which shows that we can pay bidders to be truthful. However, we may want non-negative prices:

Definition 2.7 *Clark pivot rule*
We say that we've applied the Clark pivot rule *if*

$$\forall i \in N \quad h_i(b_{-i}) = \max_{(S_j)_{j\neq i}\ alloc.} \sum_{j\neq i} b_j(S_j)$$

and then

$$\forall i \in N \quad p_i(\bar{b}) = \max_{(S_j)_{j\neq i}\ alloc.} \left(\sum_{j\neq i} b_j(S_j)\right) - \sum_{j\neq i} b_j(S_j^*)$$

Remark 2.8 *Interpretation*
We can see $p_i(\bar{b})$ as the damage caused by i to other bidders by his presence. Further, if we write

$$p_i(\bar{b}) = b_i(S_i^*) - \left[\sum_{j=1}^n b_j(S_j^*) - \max_{(S_j)_{j\neq i} alloc.} \sum_{j\neq i} b_j(S_j)\right]$$

i's payment appears as its bid for S_i^* minus a discount term which is the "value" it increased by its bid.

In the single-item auction, the discount term for player i is zero for losers and the difference between the first and second bids for the winner.

Proposition 2.9 *Non-negative payments*
Clark pivot rule payments are non-negative

Proof: $(S_j^*)_{j\neq i}$ is an allocation of S and so $\max_{(S_j)_{j\neq i}\text{alloc.}}\left(\sum_{j\neq i} b_j(S_j)\right) \geq \sum_{j\neq i} b_j(S_j^*)$ □

Proposition 2.10 *Reasonable payments*
Using the Clark pivot rule, $\forall i \in N\, p_i(S_i^*) \leq b_i(S_i^*)$. In other words, players pay less than they bid.

Proof:

$$b_i(S_i) - p_i(\bar{b}) = \sum_{i=1}^{n} b_i(S_i^*) - \max_{(S_j)_{j\neq i}\text{alloc.}} \sum_{j\neq i} b_j(S_j) = \max_{(S_j)_{j\in N}\text{alloc.}} \sum_{j=1}^{n} b_j(S_j) - \max_{(S_j)_{j\neq i}\text{alloc.}} \sum_{j\neq i} b_j(S_j) \geq 0$$

□

Corollary 2.11 *Truthfulness yields positive utilities*
If $b_i = v_i$ then $u_i(\bar{b}) \geq 0$. In other words, a player receives a positive utility from bidding truthfully.

To sum up, we've studied the VCG mechanism for combinatorial allocation under the assumptions of no collusion, private valuations and quasi-linear utilities. The VCG mechanism is truthful, it maximizes total welfare and the prices are positive with the Clark pivot rule.

3 Examples of VCG mechanisms

3.1 Single-item auctions

As we saw in remark 2.8, the VCG mechanism boils down to the Vickery auction in the single-item case.

3.2 Auctions of identical items

If we auction $m \leq n$ identical copies of an item and assuming that players only want one ($v_i(A) = \alpha_i$ if $|A| \geq 1$ and $v_i(\emptyset) = 0$), the m items are allocated to the m highest bidders at the $(m+1)$-st highest price. **Proof:** Left as an exercise to the reader. □

3.3 Procurement auctions

Assume the auctioneer wants to procure (buy) an item available with each of the n bidders (called "contractors"). Social welfare maximization is equivalent to procuring the item from the bidder with the lowest cost. The VCG mechanism will procure this item and award a payment equal to the second-lowest bidder's cost.

To formalize this, define $v_i(\emptyset) = 0$ and $v_i(1) = -c_i$ where c_i is the cost to procure from i. The VCG mechanism will award the item to the highest bidder, that is, the one with the lowest cost. Its payment is the next highest bidder's value, that is the second-lowest cost.

3.4 Public project

Suppose a city wants to build a bridge whose cost is C. There are n people who may benefit from or be hurt by the bridge. Each person i has a value $v_i \in \mathbb{R}$ if the bridge is built. We wish to build the bridge if $\sum_i v_i \geq C$ (i.e. if social welfare is positive). We cannot directly ask people what their value is because they have an incentive to lie. The VCG mechanism can be used to check if building the bridge is socially efficient.

We can prove that as a result of the VCG mechanism, if $v_i \geq 0$ then i will pay a non-zero amount only if he is *critical*. This means that $\sum_{j \neq i} v_j \leq C < \sum_j v_j$, and he will then have to pay $p_i = C - \sum_{j \neq i} v_j$. If $v_i < 0$ then i will pay a non-zero amount only if he is *critical*. In this case, being critical means that $\sum_j v_j \leq C < \sum_{j \neq i} v_j$, in which case he pays $p_i = \sum_{j \neq i} v_j - C$.

We leave this as an exercise to the reader (hint: add a player to model cost C).

3.5 Paths in a network

Suppose we want to procure a path in a network $G = (V, E)$. Each edge e is a player and has cost c_e to provide e for use. The auctioneer wants to buy a path from $s \in V$ to $t \in V$. A path is socially efficient if it minimizes the path length. The VCG mechanism then buys the cheapest path according to the lengths c_e.

We can prove that if e is not in the path P, $p_e = 0$. If $e \in P$ then the payment *to* e is equal to the benefit e brings to P when he compares the path to the cheapest path Q_e in $G - e$ ($p_e = -(c(P) - c(Q_e))$).