CS 573: Algorithmic Game Theory	Lecture date: Feb 20th, 2008
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Contents

1 Algorithmic Mechanism Design and Auctions

We saw definition of strategic games, defined the notion of equilibria and studied PoA and PoS for several games. However, in those game with *predefined rules*, what we could do was only to see what kind of behavior arose with strategic players. In "Mechanism Design," the problem is *inverse*. We *make rules* in order to induce certain *desired behaviors* from the agents behaving selfishly and independently.

Mechanism Design has close connections to auction theory since it primarily arose in that context. In computer science, we focus on "*Algorithmic*" Mechanism Design in the following sense.

- computational aspects such as tractability in terms of computational time and communication will be considered as important
- the new point would also be somewhat worst-case oriented. Economists are primarily interested in the Bayesian(distributional) setting.

Read Chapter 9 for more details.

1.1 Auctions

Auctions have a long history and have been an important part of economic activity since antiquity. They have gained prominence in economics since the landmarking work of Vickrey and others who put their formal study on a sound mathematical and game theoretic footing. Also, they have gained prominence in a practical sense due to the privatization of public assets on a large scale.
There are many types and models for auctions and we will only cover a few. See the references on the course webpage for more reading.

1.1.1 Single Item Private Value Auctions

Let us state with a simple setting

- single item to be sold.
- n bidders/players
- each player i has *private value* $v_i (\geq 0)$ for the item

Figure 1: relation between auctions

By private value we mean that player i's value v_i does not depend on what other players' values are for the item. This is an important assumption and is not valid for all items. It is most suitable for items that are to be consumed privately by each player (ex. art works, though things will be different if we think of resale)

We consider four types of auctions for this setting

1. Dutch or Descending Auction
   - price starts at infinity and comes down
   - first player to raise hand gets the item at current price
2. English or Ascending Auction
   - price starts at zero and goes up
   - bidders drop and once they do, cannot come in
   - last person to remain gets the item at a price just above the price at which the previous person drops out
3. Sealed-bid First Price Auction
   - all bidders submit their bids to auctioneer in a sealed envelope
   - highest bidder wins and pays his/her bid
4. Sealed-bid Second Price Auction
   - all bidders submit their bids to auctioneer in a sealed envelope
   - highest bidder wins and pays the second highest bid

We can see a rough equivalence between the English and second-price auction and between the Dutch and first-price auction,

One can informally see that the bidding strategies are the same for a player in the Dutch and First-price auctions. In both auctions, no information about losing bidders is revealed. Moreover, although the Dutch auction is an open auction, it ends as soon as the winner is determined. (It can be shown more formally that the two auctions are "strategically equivalent".)
The equivalence between English and second-price auction is less strong. The bidding strategies for a player are the same in both forms but this requires the strong private value assumption. This is because the ascending auction reveals information about losing bidder's valuations.
Vickrey formalized the sealed-bid second-price auction and analyzed its strategic and game theoretic properties and hence it is also often called the *Vickrey auction.*(he won the Nobel prize for his auction work.)
We will now prove several simple but important properties of the second-price auction under the following setting.

- b_i: bid of player of i
- v_i: value of player i for item (not necessarily $b_i = v_i$)
- p_i: payment of player i
- $u_i = v_i - p_i$: utility of player i

Assumption 1.1 (for second-price auction)

1. *each bidder has a quasi-linear utility function wants to maximize $u_i = v_i - p_i$*
2. *No collusion among players*
3. *valuations are private (v_i is not influenced by v_j)*

The following lemma is the most important property of Vickrey auction.

Lemma 1.2 *For each player i and for set of bids $\{b_j\}_{j\neq i}$ of players other than i, utility maximizing bid for i is $b_i = v_i$*

Proof: Fix i and bids $\{b_j\}_{j\neq i}$. Let $B = \max_{j\neq i} b_j$. Consider the following cases.

1. $v_i < B$: For $b_i = v_i$, $u_i = 0$, since i does not win and pays nothing. Suppose $b_i \neq v_i$.
   (a) $b_i \leq B$: $u_i = 0$
   (b) $b_i > B$: $u_i = v_i - B < 0$, since i wins and pays B

   So $b_i = v_i$ is utility maximizing
2. $v_i > B$: If $b_i = v_i$, $u_i = v_i - B > 0$. If $b_i \neq v_i$
   (a) $b_i \leq B$: $u_i = 0$
   (b) $B < b_i < v_i$ or $v_i < b_i$: $u_i = v_i - B$
3. $v_i = B$: If $b_i = v_i$, this case depends on auction rule. If i wins, i pays second highest B. So $u_i = v_i - B = 0$. If the other wins, i pays nothing, still $u_i = 0$. If $b_i \neq v_i$,
   (a) $b_i < v_i = B$: i loses, $u_i = 0$
   (b) $b_i > v_i = B$: i wins, $u_i = v_i - B = 0$

□

Note that is i's strategy will remain the same whether i knows B or not. This relies on private value assumption. So i's utility does not get influenced by knowledge of B.

Lemma says that for *rational* players *a* best strategy is to bid *truthfully* i.e. $b_i = v_i$. The mechanism is said to be *strategy proof*. In game theory words, $b_i = v_i$ is a "*dominant*" strategy for i.

Definition 1.3 (Dominant strategy) *A strategy $s_i \in S_i$ is a dominant strategy if $u_i(s_i, s_{-i}) = \max_{a \in S_i} u_i(a, s_{-i})$ for all s_{-i} i.e. choosing s_i is a utility maximizing strategy whatever the other players do.*

Definition 1.4 (Strongly Dominant strategy) *A strategy $s_i \in S_i$ is a strongly dominant strategy if s_i is dominant and $\exists s_{-i}$ s.t. $\forall s_i^{'} \neq s_i$ $u_i(s_i, s_{-i}) > u_i(s_i^{'}, s_{-i})$*

Definition 1.5 (Weakly Dominant strategy) *Dominant but not strongly dominant strategy*

Now we prove that bidding $b_i = v_i$ is a dominant strategy for each player i.

Lemma 1.6 *For each player i, if $b_i \neq v_i$, then there is a set of bids $\{b_j\}_{j \neq i}$ s.t. utility of i is less than that of bidding v_i.*

Proof: Suppose $b_i \neq v_i$. Set $b_j = \frac{b_i + v_i}{2}$ for $\forall j \neq i$.

1. $b_i < v_i$: i loses and $u_i = 0$, while bidding v_i would get positive utility equal to $v_i - \frac{b_i + v_i}{2}$
2. $b_i > v_i$: $u_i = v_i - \frac{b_i + v_i}{2} < 0$, while bidding v_i would yield utility 0.

$\square$

Proposition 1.7 *Truthful bidders receive non-negative utility.*

A mechanism with above property is said to have the *individually rational* property. Also sometimes referred to as having *voluntary participation* property.

Proposition 1.8 *Vickrey auction maximizes the **social welfare**. i.e. Vickrey auction allocates the item to the player that values it the most.*

A mechanism is called *efficient* if it maximizes the social welfare of the allocation. The social welfare objective which is also called the *utilitarian* objective sometimes will become clearer in the context of combinatorial auctions and the generalization of Vickrey auction mechanism to the VCG mechanism.
Vickrey auction can be implemented in polynomial time and works with generalized valuation.(no assumption on the range of the numbers)

1.2 Mechanism design and Payments

One can view Vickrey Auction as a mechanism to allocate the item to the player who values it the most. To achieve it, we charged bidders with some money. The prices were basically a "way" to enforce truthfulness which results in the item being given to the one who values it the most. But can we make the goal without payments? In general, the answer is No because of impossibility results of Arrow, Gibbard and Satterthwaite.

Before studying Arrow's results on elections, it is worth asking why a simple majority voting rule commonly used can be problematic. If we have only two candidates, there is no problem. But with three candidates, things become not that simple. Consider three candidates a, b, and c and three voters with the following preferences- $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$($a \succ_1 b$ means that voter i perfers a to b). A majority (1 and 3) prefers a to b, 1 and 2 prefers b to c and 2, 3 prefers c to a. So in the joint majority choice, we have $a \succ b \succ c \succ a$, which is not consistent. This example is called as Condorcet's Paradox.

This tells us we need more complicated voting method for a desirable social choice. But it turns out that it is impossible, whatever voting rule is adopted. To describe Arrow's theorem, we need the following notations and definitions.

- A: Set of candidates
- L: Set of all preference relation over A (aka permutation)
- $f : L^n \to L$: A voting rule called as social welfare function

Definition 1.9

1. *Unanimity:* $f(\prec, \prec, \cdots, \prec) = \prec$
2. *f is a dictator to i if* $f(\prec_1, \prec_2, ..., \prec_i, ..., \prec_n) = \prec_i$
3. *f is said to satisfy independence of irrelevant alternatives if $\forall a, b \in A$, f's ordering of a, b depends only on how each player orders a, b. Formally, for $\forall a, b \in A$ and every $\prec_1, ..., \prec_n, \prec_1^{'}, ... \prec_n^{'} \in L$, if we denote $\prec = f(\prec_1, ... \prec_n)$ and $\prec^{'} = f(\prec_1^{'}, ... \prec_n^{'})$ then $a \prec_i b \Leftrightarrow a \prec_i^{'} b$ for all i implies that $a \prec b \Leftrightarrow a \prec^{'} b$.*

Theorem 1.10 (Arrow) *If $|A| \geq 3$, then any welfare function satisfying unanimity and independence of irrelevant alternatives is a dictatorship*