Homework 2
Spring 2008, CS 573: Algorithmic Game Theory

Instructions and Policy: You are allowed to use the textbook, the class notes and lecture notes from other courses pointed to in the webpage. However, you are explicitly *not* allowed to consult published research papers on the topics related to the problems.

You can discuss with other students (ideally with not more than two or three other students) in obtaining a solution but each student has to write up the solutions separately in their own words.

Please write clearly and concisely.

The homework will be graded on a good/fair/bad/ugly scale.

Problem 1 Consider the Public Project example (Section 9.3.5.5) in the text book. Define the setting formally as a social welfare maximization allocation problem and prove the claims made in the paragraph regarding payments.

Problem 2 Read Sections 12.1 and 12.2 from the text book. Solve Problem 12.2.

Problem 3 Multi-unit auctions. There are m identical items to be auctioned off to n players. Each player is single-minded and wants q_i items at a value v_i (more that q_i items give the same value). Since the bidders are single-minded the bids are of the form (p_i, b_i) where p_i is an integer in $1, \ldots, m$ that indicates the number of items of interest and b_i is the bid for them (assume b_i are rational numbers).

- Assuming m is given in binary, prove that the winner determination problem is NP-hard by reducing the knapsack problem to it. The knapsack problem is the following: given a knapsack of capacity B and n items with item i having a size s_i and profit p_i, find a maximim profit subset of items that fit into the knapsack (assume all numbers are integer valued).
- Develop a simple greedy type algorithm for the knapsack problem that gives a $1/2$-approximation.
- Use the above greedy algorithm to devise a polynomial time truthful mechanism for the multi-unit auction problem that achieves a $1/2$ approximation for the social welfare. Note that the players can cheat both in the number of items they request as well as their bid.

Note: There is a classical fully polynomial time approximation scheme (FPTAS) for knapsack; for every $\epsilon > 0$ the algorithm yields a $(1-\epsilon)$ approximation in time polynomial in n and $1/\epsilon$. However, this classical FPTAS is not monotone and cannot be used in a truthful mechanism. Recent work, motivated by game theoretic considerations, has led to the development of a new monotone FPTAS that can be used in a truthful mechanism.

Problem 4 Let $T = (V, E)$ be a capacitated tree with u_e representing an integer capacity for each $e \in E$. There are n players and player i is interested in obtaining a path between nodes s_i and t_i to route one unit of demand. Player i has a private value v_i for obtaining a path. The end points s_i, t_i for the players are public knowledge and hence this is a single-parameter problem. Paths for different players have to share edge capacities; for an edge e the paths of at most u_e players can contain it. The goal is to design a polynomial time truthful mechanism to approximately maximize social welfare. The winner determination problem is to solve the following integer program where x_i is the variable which is 1 if i is assigned a path and 0 otherwise. Let $P(s_i, t_i)$ denote the unique path between s_i and t_i in T.

$$\max \sum_i v_i x_i$$

$$\sum_{i: e \in P(s_i,t_i)} x_i \quad \leq \quad u_e \qquad e \in E$$

$$x_i \quad \in \quad \{0,1\} \qquad 1 \leq i \leq n$$

It is known that the above optimization problem is NP-hard. Here we develop a monotone approximation algorithm.

- Consider the following approximation algorithm. Root T at some arbitrary node r. For each player i with (s_i, t_i) define $\ell(i)$ to be the depth of the least common ancestor of s_i and t_i. Reorder the players such that $\ell(1) \geq \ell(2) \geq \ldots \geq \ell(n)$. The algorithm iteratively goes over the players one by one in this order and assigns a path to player i if it can feasibly do so given the previously assigned paths. This algorithm is known to provide a $1/2$ approximation when $v_i = 1$ for all i. Prove this when $u_e = 1$ for all e. **Extra Credit:** Prove this fact for arbitrary u_e. You can use the more general fact in subsequent parts even if you don't prove it.

- When v_i can be different consider the following algorithm. Let $v_{\max} = \max_i v_i$ and $v_{\min} = \min_i v_i$. Let $\alpha = v_{\max}$. For $j = 0, 1, 2, \ldots$, let $S_j = \{i \mid v_i \in [\alpha/2^j, \alpha/2^{j+1})\}$. Note that at most n of these sets are non-empty. For each non-empty S_j run the approximation algorithm from the previous step by pretending that the value of each pair is 1. Let $S'_j \subseteq S_j$ be the set of pairs routed by the algorithm from S_j. Output the solution S'_k where $k = \text{argmax}_j |S'_j| \alpha/2^{j+1}$. Show that this yields an $\Omega(1/\log(\min\{n, v_{\max}/v_{\min}\}))$ approximation.

- Show that the approximation derived in the previous step is not a *monotone* algorithm. Devise a modified algorithm with a similar approximation ratio that is monotone. *Hint:* Slightly alter the the grouping scheme.

- Using the monotone algorithm, give a polynomial time truthful mechanism for the problem that provides a $\Omega(1/\log(\min\{n, v_{\max}/v_{\min}\}))$ approximation to social welfare.

- **Open Problem:** Is there a polynomial time truthful mechanism that achieves a constant factor approximation to the social welfare? There is a polynomial time $1/4$ approximation algorithm for the winner determination problem (with arbitrary v_i), however the algorithm is not monotone.