Homework 1
Spring 2008, CS 573: Algorithmic Game Theory

Instructions and Policy: You are allowed to use the textbook, the class notes and lecture notes from other courses pointed to in the webpage. However, you are explicitly *not* allowed to consult published research papers on the topics related to the problems.

You can discuss with other students (ideally with not more than two or three other students) in obtaining a solution but each student has to write up the solutions separately in their own words.

Please write clearly and concisely.

The homework will be graded on a good/fair/bad/ugly scale.

Problem 1 Prove that the theorem that gives a sufficient condition for the existence of pure Nash equilibria in a finite game (see page 2 of scribed notes from 1/18/08) impiles Nash's theorem on existence of mixed equilibria.

Problem 2 In class we considered the price of anarchy of load balancing games. Here we consider the following setting. There are n jobs that wish to be scheduled on m machines. Each job has a weight w_i and it can only be scheduled on a specified set $S_i \subseteq \{1, 2, \ldots, m\}$ of machines. Each job selfishly tries to pick the machine (from its set of allowed machines) with the least load. The load of a machine is the sum of the weights of the jobs that select it. The social cost is makespan, the maximum machine load. Show that the price of anarchy in this game is $O(\log m / \log \log m)$.
Hint: use an approach similar to that for the case of machines with different speeds.
Extra Credit: Give an example to show a matching lower bound.
Extra Credit: Extend the analysis to the case where machines also have different speeds.

Problem 3 Prove that the Pigou bound for non-atomic selfish routing is $4/3$ for linear edge-cost functions. Extend this to prove the same bound for concave edge-cost functions.

Problem 4 In the second proof of the Pigou bound optimality we considered a new graph G' obtained from G by creating a parallel copy e^* of each edge e in G and setting the cost function on e^* to be the constant function with value $c_e(f_e)$ where f is an equilibrium flow. We then showed the following: if f_e is split between e and e^* optimally so as to minimize the cost, the resulting flow on e is x_e (and $f_e - x_e$ on e^*) where x_e is a soltuion to the equation:

$$c_e(x_e) + x_e c'_e(x) = c_e(f_e).$$

In the above, $c'_e(x)$ is the derivative of $c_e(x)$. Show that the flow f^* in G' defined by $f^*_e = x_e$ and $f^*_{e^*} = f_e - x_e$ is an equilibrium flow in G' for the edge-cost functions defined by $\bar{c}_e(x) = c_e(x) + x \cdot c'_e(x)$. Use this to conclude that f^* is an optimum flow in G'. Why does this lead to an upper bound on the price of anarchy that matches the Pigou bound?

Problem 5 Solve Problem 18.4 from the textbook. Is the potential functions an *exact* potential function?