Local Search

Given an instance I of problem Π

S(I): space of feasible solutions for I

for state s, define N(s) as *local neighbourhood of s*

typically |N(s)| is *polynomial* in input size |I|

or

there is a polynomial time algorithm (oracle) to find best solution in N(s)

Local Search

start with some initial state $s_0 \in S$

s_i: state in iteration i

Local improvement step in iteration i:

find a new state $s_{i+1} \in N(s_i)$ such that $val(s_{i+1}, I)$ is *strictly better* than $val(s_i, I)$

if no such state s_{i+1} then STOP and output *local optimum* s_i

Local Search

Observation: for problems where local optimum implies global optimum (convex optimization problems) local search yields an optimum solution

Observation: convergence time not necessarily strongly polynomial in the input size

Modified Local Search

For minimization problems

STOP if $val(s_{i+1}) \leq (1+\varepsilon)\ val(s_i)$

Guaranteed to stop in $O(\log\ (val(s_0/OPT)\ /\varepsilon)$ iterations. Usually one can choose s_0 s.t $val(s_0)/OPT \leq poly(|I|)$

Local search in approximation

we need to define local neighbourhood for states

given s, finding $s' \in N(s)$ s.t $val(s') < val(s)$ should be poly-time computable

we need to argue that a local optimum is within an α factor of a global optimum

α would be the approximation ratio

(Facility) Location Problems

F : set of locations for facilities that can serve clients

D : set of clients/demands that need service

Typical problem:

- *open* some facilities in F
- *assign* clients in D to facilities in F

Goal: minimize total cost (of facilities and clients)

Uncapacitated Facility Location

F : set of locations for facilities

D : set of clients

f_i : cost to open a facility at $i \in F$

c_{ij} : cost/distance of serving client j from facility i

Feasible solution: $S \subseteq F$ with $|S| \geq 1$

and an assignment $\sigma: D \rightarrow F$

Uncapacitated Facility Location

Feasible solution: $S \subseteq F$ with $|S| \geq 1$

and an assignment $\sigma: D \rightarrow F$

S : opened facilities

$\mathrm{cost}(S,\sigma) = \sum_{i \in S} f_i + \sum_{j \in D} c_{\sigma(j), j}$

$\sum_{i \in S} f_i$: facility cost

$\sum_{j \in D} c_{\sigma(j), j}$: service/shipping cost

Goal: find S, σ with minimum cost

Uncapacitated Facility Location

S : opened facilities

$\text{cost}(S,\sigma) = \sum_{i \in S} f_i + \sum_{j \in D} c_{\sigma(j), j}$

$\sum_{\iota \in \Sigma} f_i$: facility cost

$\sum_{j \in D} c_{\sigma(j), j}$: service/shipping cost

Given S, σ is defined: $\sigma(j) = \text{argmin}_i\, c_{i,j}$

Metric assumption

Assume c satisfies triangle inequality

that is, F and D are points in a metric space and c_{ij} is the distance from i to j

Exercise: show that if c does *not* satisfy triangle inequality then problem is at least as hard as the set cover problem

Local Search for UFL

Solution specified by a set of facilities (σ implicit)

Three natural *moves*

Given current solution $S \subseteq F$

- *add* a new facility k in F - S
- *delete* a current facility i in S
- *substitute* a current facility i in S with a new facility k in F-S

Local Search for UFL

Given current solution $S \subseteq F$

- *add* a new facility k in $F - S$
- *delete* a current facility i in S
- *substitute* a current facility i in S with a new facility k in $F-S$

Can show that local optimum with above moves is a 5-approximation

Slightly different local search

Current solution (S, σ)

for each $i \in F$ compute $S' = Move(S,i)$ as follows:

- add i to S if it is not already in S
- $X = \{ j \mid c_{\sigma(j),j} > c_{ij} \}$
- for each i' in $S - \{i\}$ do
  - *remove* i' and assign clients in $\sigma^{-1}(i') – X$ to i if this reduces cost

Slightly different local search

gain(S,i) = cost(S) – cost(S'=Move(S,i))
(could be negative)

if gain(i) > ε cost(S)/4n then change S to S'
otherwise STOP and output S

Theorem: Output of algorithm is a (3+ε) approximate solution

Bounding # of iterations

Let S_0 be the starting solution

Algorithm stops in

$O(1/\varepsilon\ n^2 \log n \log \text{cost}(S_0)/OPT)$ iterations

Follows from the fact that

$\text{cost}(S_{i+1}) \leq (1-\varepsilon/4n)\ \text{cost}(S_i)$

Initial solution

Sort facilities s.t $f_1 \leq f_2 \leq \ldots \leq f_n$

Consider $X_i = \{1,2,\ldots,i\}$ (first i facilities)

$i = \text{argmin}_j \text{ cost}(X_j)$

initial solution $S_0 = X_i$

Lemma: $\text{cost}(S_0) \leq |F| \text{ OPT}$

Proof: exercise

Analysis

$\text{cost}(S)$ = facility cost + service cost

$\text{cost}_f(S)$ = facility cost of S = $\sum_{i \in S} f_i$

$\text{cost}_s(S)$ = service cost of S = $\sum_{j \in D} c_{\sigma(j), j}$

We assume σ is implict from S

Analysis

$gain(S,i)$: gain from moving from S to S' = $Move(S,i)$

Note: $gain(S,i)$ could be negative

Let O be any solution (say optimum soln)

Lemma 1: $\sum_{i \in O} gain(S, i) \geq cost_s(S) - cost(O)$

Lemma 2: $\sum_{i \in O} gain(S, i) \geq cost_f(S) - 2cost(O)$

Analysis

Lemma 1: $\sum_{i \in O} gain(S, i) \geq cost_s(S) - cost(O)$

Lemma 2: $\sum_{i \in O} gain(S, i) \geq cost_f(S) - 2cost(O)$

Lemma 1 useful to bound service cost of S

Lemma 2 useful to bound facility cost of S

Use above lemmas to prove following:

If algorithm stops with S then
$cost(S) \leq (3+\varepsilon)$ OPT

Analysis

$\text{cost}(S) = \text{cost}_f(S) + \text{cost}_s(S)$

combining Lemmas 1 and 2

$2 \sum_{i \in O} \text{gain}(S,i) \leq \text{cost}_f(S) + \text{cost}_s(S) - 3\text{cost}(O)$

implies

$\text{cost}(S) - 2 \sum_{i \in O} \text{gain}(S,i) \leq 3\ \text{OPT}$

Since S is a near-local optimum

$\text{gain}(S,i) \leq \varepsilon/4n\ \text{cost}(S)$ for any $i \in F$

Analysis

$\text{cost}(S) - 2\sum_{i\in O} \text{gain}(S,i) \leq 3\ \text{OPT}$

Since S is a near-local optimum
$\text{gain}(S,i) \leq \varepsilon/4n\ \text{cost}(S)$ for any $i \in F$

Since $|O| \leq n$ we have

$\text{cost}(S)(1 - \varepsilon/2) \leq 3\text{OPT}$

hence
$\text{cost}(S) \leq (3+\varepsilon)\ \text{OPT}$ for sufficiently small ε

Proofs of Lemma 1,2

Solutions S, O

let σ, ω denote the assignments of clients to facilities in the two solutions

Proof of Lemma 1

gain'(S,i) : cost reduction if we add i to S but did not remove any facilities in S

(since in Lemma 1 we only care about service cost)

clearly gain'(S, i) $\leq$ gain(S,i)

consider $\sum_{i \in O}$ gain'(S,i)

The only reduction in cost happens because clients get reassigned

Proof of Lemma 1

consider $\sum_{i \in O}$ gain'(S,i)

The only reduction in cost happens because clients get reassigned

Consider client j: it contributes at least

$c_{\sigma(j), j} - c_{\omega(j),j}$

therefore total contribution of all clients is at least

$\sum_{j \in D} (c_{\sigma(j), j} - c_{\omega(j),j}) \geq \text{cost}_s(S) - \text{cost}_s(O)$

Increase in cost is due to facilities in O

therefore total gain

$\geq \text{cost}_s(S) - \text{cost}_s(O) - \text{cost}_f(O)$

Proof of Lemma 2

for each k in S let $\pi(k)$ be the closest facility in O ($\pi(k) = k$ if $k \in O$)

$gain'(S,i)$: cost reduction if we add i to S and removed all facilities in $\pi^{-1}(i)$ and transferred their clients to i

$gain'(S,i) \leq gain(S, i)$

Proof of Lemma 2

for each k in S let $\pi(k)$ be the closest facility in O ($\pi(k) = k$ if $k \in O$)

$\sum_{\iota \in O}$ gain'(S,i) ??

facility cost gain = $cost_f(S) - cost_f(O)$

each facility of S is removed and each facility of O is added

Proof of Lemma 2

service cost reduction ?

consider client j:

let $k = \sigma(j)$ for ease of notation

if $i = \omega(j)$ then contribution of j in $gain'(S,i)$ is
$c_{k,j} - c_{i,j}$

same when $i = \pi(k)$

for other i gain can only be positive and we ignore

Proof of Lemma 2

if $i = \omega(j)$ then contribution of j in $gain'(S,i)$ is

$c_{k,j} - c_{i,j}$

same when $i = \pi(k)$

however it could be the case that $\omega(j) = \pi(k)$

we ensure that we do not over count

Proof of Lemma 2

Consider the case $i = \pi(k)$

Claim: $c_{k,j} - c_{i,j} \geq -(c_{k,j} + c_{\omega(j),j})$

Note rhs is always negative so we can add the gain for $i = \pi(k)$ with gain for $i = \omega(j)$ without any problem so total gain contribution of client j

$\geq c_{k,j} - c_{\omega(j),j} - (c_{k,j} + c_{\omega(j),j}) \geq -2\, c_{\omega(j),j}$

Proof of Lemma 2

$\sum_{\iota \in O}$ gain'(S,i)

$\geq$ facility cost gain + service cost gain

$\geq$ $cost_f(S) - cost_f(O) - \sum_j 2c_{\omega(j), j}$

$\geq$ $cost_f(S) - cost_f(O) - 2cost_s(O)$

Proof of Claim

Consider the case $i = \pi(k)$

Claim: $c_{k,j} - c_{i,j} \geq -(c_{k,j} + c_{\omega(j),j})$

(see pic in next slide)

$c_{i,j} \leq c_{k,j} + c_{k,i}$ (trianlge ineq)

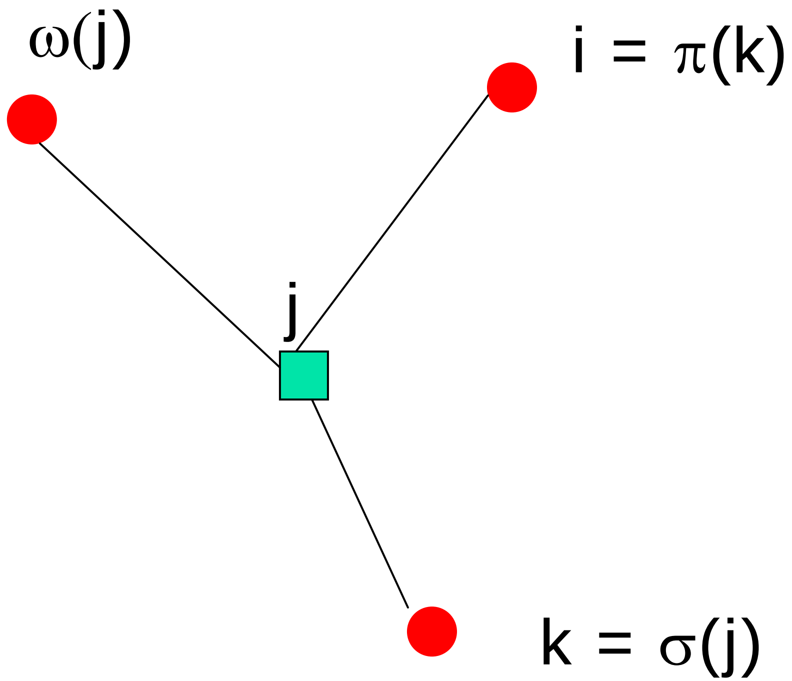
but since $i = \pi(k)$, i is the closest facility to k implies $c_{k,i} \leq c_{k,\omega(j)}$ therefore

$c_{i,j} \leq c_{k,j} + c_{\omega(j),i}$

$\leq c_{k,j} + c_{\omega(j),j} + c_{k,j}$ (trianlge ineq again)

$\leq 2c_{k,j} + c_{\omega(j),j}$

Proof of Claim



Improved algorithm

Recall:

Lemma 1: $\sum_{i \in O} \text{gain}(S,i) \leq \text{cost}_s(S) - \text{cost}_f(O) - \text{cost}_s(O)$

Lemma 2: $\sum_{i \in O} \text{gain}(S,i) \leq \text{cost}_f(S) - \text{cost}_f(O) - 2\text{cost}_s(O)$

Asymmetry in Lemma 2 can be exploited

Scaling

- Pretend facilities are more expensive than they are: multiply facility costs by a factor α

- For a solution X let cost(X) and cost'(X) denote costs with original and modified facility cost

- Run local search on modified costs to find soln S
- Output S for the original problem
- Let O be an optimum solution to original problem

Analysis

From local search analysis

$cost'_s(S) \leq cost'_f(O) + cost'_s(O) + \varepsilon/4\ cost'(S)$

and

$cost'\_f(S) \leq cost'_f(O) + 2cost'_s(O) + \varepsilon/4\ cost'(S)$

$cost(S) = cost'_s(S) + 1/\alpha\ cost'_f(S)$

$\leq \alpha\ cost_f(O) + cost_s(O) + cost_f(O) + 2/\alpha\ cost_s(O) + \varepsilon/4\ cost(S)\ (\alpha + 1)$

Analysis

$\text{cost}(S) \leq \alpha\ \text{cost}_f(O) + \text{cost}_s(O) + \text{cost}_f(O) + 2/\alpha\ \text{cost}_s(O) + \varepsilon/4\ \text{cost}(S)\ (\alpha + 1)$

$\leq (\alpha+1)\text{cost}_f(O) + (1+2/\alpha)\text{cost}_s(O) + \varepsilon\ (\alpha+1)/4\ \text{cost}(S)$

choosing α = sqrt{2} and ε sufficiently small we get that

$\text{cost}(S) \leq (1+\text{sqrt}\{2\} + \varepsilon')\ \text{cost}(O)$