Multiway Cut Problem

Given undirected graph $G = (V, E)$

Edge weights $w: E \rightarrow \mathcal{R}^+$

Terminals: $T = \{t_1, t_2, \ldots, t_k\} \subseteq V$

**Goal:** find minimum weight set of edges $E'$ such removing $E'$ separates all terminals

That is, no connected component of $G(V,E-E')$ has two terminals from $T$
Facts

\( k = 2 \), standard s-t cut problem, polynomial time solvable

Multiway Cut is NP-hard and also APX-hard even for \( k = 3 \)

Multiway Cut can be solved exactly for fixed \( k \) in planar graphs
Isolating Cuts

For $t_i \in T$, $E' \subseteq E$ is an isolating cut if removing edges in $E'$ separates $t_i$ from all other terminals.

Minimum weight isolating cut can be computed in polynomial time (How?)

Let $C_1, C_2, ..., C_k$ be min weight isolating cuts for $t_1, t_2, ..., t_k$. 
Isolating Cut Heuristic

Let $C_1, C_2, \ldots, C_k$ be min weight isolating cuts for $t_1, t_2, \ldots, t_k$

Assume wlog that $w(C_1) \leq w(C_2) \leq \ldots \leq w(C_k)$

Output $C = C_1 \cup C_2 \cup \ldots \cup C_{k-1}$

Claim: $C$ is a feasible multiway cut
Analysis

Theorem: \( w(C) \leq 2(1-1/k) \text{OPT} \)

Consider some optimum cut \( A \)
Let \( G[V_i] \) be the connected component in \( G(V,E-A) \)
that contains terminal \( t_i \)

\[ A_i = \delta(V_i) : \text{edges with exactly one end point in } V_i \]
Analysis

Consider some optimum cut A
Let \( G[V_i] \) be the connected component in \( G(V,E-A) \) that contains terminal \( t_i \)
\( A_i = \delta(V_i) : \) edges with exactly one end point in \( V_i \)

Claim: \( 2w(A) = \sum_{i=1}^{k} w(A_i) \)

Each edge in \( A \) is counted twice
$A$ and $A_1$, $A_2$, ..., $A_k$

e counted in $A_1$ and $A_4$
Analysis

Claim: \(2\text{OPT} = \sum_{i=1}^{k} w(A_i)\)

\(A_i\) is an isolating cut for \(t_i\) therefore for \(1 \leq i \leq k\)  
\(w(A_i) \geq w(C_i)\) since \(C_i\) was min-wt isolating cut for \(t_i\)  
implies  
\[\sum_{i=1}^{k} w(A_i) \geq \sum_{i=1}^{k} w(C_i)\]
Analysis

\[ w(C) = w(C_1) + \ldots + w(C_{k-1}) \]

\[ \leq (1 - \frac{1}{k}) (w(C_1) + \ldots + w(C_k)) \quad \text{(since } C_k \text{ is the heaviest cut)} \]

\[ \leq (1 - \frac{1}{k}) (w(A_1) + \ldots + w(A_k)) \]

\[ \leq 2(1 - \frac{1}{k}) \, w(A) \]
A Tight Example

Blue: opt cut of value 3

Yellow: algorithm’s cut of value 4-2ε

Example can be generalized to large k to reach ratio of (2-2/k)
A Greedy Splitting Algorithm

Start with G
Split into two components such that each contains a terminal
Split *one* of the two components such that each of the three components has a terminal
Split *one* of the three components such that each of the four components has a terminal
... till k components each with terminal

*At each step choose a cheapest cut among components*
Greedy Splitting Algorithm

**Theorem:** Greedy splitting also a $2 - \frac{2}{k}$ approximation algorithm for multiway cut

**Proof:** Exercise
The k-Cut Problem

Given undirected $G=(V,E)$

$w: E \rightarrow \mathcal{R}^+$

integer $k$

Goal: find minimum weight set of edges to remove such that $G$ is partitioned into $k$ connected components
Facts

\( k=2 \) is the global mincut problem, can be solved in polynomial time (near linear time using randomization)

Can be solved in \( O(n^{k^2}) \) time – hence polynomial time solvable for fixed \( k \)

NP-hard for arbitrary \( k \)
A Greedy Splitting Algorithm

Start with G
Split into two components
Split into three components by splitting one of the two components

...

Choose cheapest split at each stage
Greedy Splitting

Theorem: Greedy splitting is a $2 - \frac{2}{k}$ approximation

Proof is complicated

We do an alternate proof using Gomory-Hu trees
Cut Structure of Undirected Graphs

Given undirected graph $G = (V, E)$
$w: E \rightarrow \mathbb{R}^+$

Let $mc(ab)$ denote weight of min $a$-$b$ cut in $G$

There are $n(n-1)/2$ pairs of vertices so potentially $n(n-1)/2$ different min-cut values

However ...
Cut Structure of Undirected Graphs

There are $n(n-1)/2$ pairs of vertices so potentially $n(n-1)/2$ different min-cut values. However only $n-1$ distinct cut-values.

Moreover magical Gomory-Hu tree!
Gomory-Hu tree for \( G \)

\( G = (V, E) \) with edge weights \( w \)

Gomory-Hu tree \( T = (V, E_T) \)

\( u : E_T \rightarrow \mathbb{R}^+ \)

same vertex set as \( G \)

\( u : \) weights on edges of \( T \)

For each pair \((a, b)\) of vertices of \( G\), their min-cut value \( mc(ab) \) in \( G \) is \textit{equal to} min cut value in \( T \)!
Gomory-Hu tree for $G$

For each pair $(a, b)$ of vertices of $G$, their min-cut value $mc(ab)$ in $G$ is *equal to* min cut value in $T$!

Min-cut in $T$ is min-weight edge in *unique* path connecting $a$ and $b$ in $T$

In particular for an edge $ab$ in $E_T$ we have $u(ab) = mc(ab)$
Gomory-Hu Tree for $G$

Can be computed using $O(n)$ s-t cut computations

With each edge $ab$ in $E_T$ we can also associate a min a-b cut $C_{ab}$ of value $u(ab)$
Removing edges in $C_{ab}$ disconnects $a$ from $b$
k-Cut alg using Gomory-Hu trees

Run Greedy Splitting on $T$ instead of $G$

Equivalent to picking the $k-1$ lightest edges in $T$

Let $e_1$, $e_2$, ..., $e_{k-1}$ be the chosen edges (from $T$)

Output $C = C_{e_1} \cup C_{e_2} \cup ... \cup C_{e_{k-1}}$
Analysis

**Claim:** Removing \( C \) results in \( k \) components

Easy exercise
Analysis

Theorem: \( w(C) \leq 2\left(1-\frac{1}{k}\right) \text{OPT} \)

Let \( A \) be an optimum cut and let \( V_1, V_2, \ldots, V_k \) be the connected components
\[ A_i = \delta(V_i) \]

As before assume wlog \( w(A_1) \leq w(A_2) \ldots \leq w(A_k) \)
and we have \( w(A_1) + \ldots + w(A_k) = 2w(A) = 2\text{OPT} \)
Analysis

Lemma: \( \sum_{i=1}^{k-1} u(e_i) \leq \sum_{i=1}^{k-1} w(A_i) \)

Assuming lemma we have

\[
\begin{align*}
  w(C) &= u(e_1) + \ldots + u(e_{k-1}) \\
  &\leq (1-1/k)(w(A_1) + \ldots + w(A_k)) \\
  &\leq 2(1-1/k) \text{ OPT}
\end{align*}
\]
Proof of Lemma

We identify distinct edges $f_1, f_2, \ldots, f_{k-1}$ of $T$ s.t $w(A_i) \geq u(f_i)$ for $1 \leq i \leq k-1$

Since algorithm picks lightest $k-1$ edges we have the lemma

Note that $f_1, \ldots, f_{k-1}$ are not necessarily related to $e_1, \ldots, e_{k-1}$
Proof of Lemma

We identify edges $f_1, f_2, \ldots, f_{k-1}$ of $T$ s.t $w(A_i) \geq u(f_i)$ for $1 \leq i \leq k-1$

Obtain tree $T'$ from $T$ as follows:
- shrink each $V_i$ to a single vertex
- throw out parallel edges between vertices in $T'$

$T'$ is connected since $T$ is connected
Proof of Lemma

Let \( t_1, t_2, \ldots, t_k \) be vertices of \( T' \) with \( t_i \) corresponding to \( V_i \).

Root \( T' \) at \( t_k \) (recall \( A_k \) was the heaviest cut).

Orient edges in \( T' \) towards the root.

Let \( f_i \) be the unique edge directed from \( t_i \) towards the root in the orientation.
Proof of Lemma

Root $T'$ at $t_k$ (recall $A_k$ was the heaviest cut)
Orient edges in $T'$ towards the root
Let $f_i$ be the unique edge out of $t_i$ in the orientation

Remark: $f_i$ is an edge of $T$

Let $f_i = ab$ where $a \in V_i$, $b \notin V_i$
Proof of Lemma

Root $T'$ at $t_k$ (recall $A_k$ was the heaviest cut)
Orient edges in $T'$ towards the root
Let $f_i$ be the unique edge out of $t_i$ in the orientation

Let $f_i = ab$ where $a \in V_i$, $b \notin V_i$

From Gomory-Hu tree property $mc(ab) = u(f_i)$
Also $A_i$ is a cut that separates $a$ from $b$ hence $w(A_i) \geq mc(ab) = u(f_i)$
Tight Example

Same as that for multiway cut
The Steiner $k$-Cut problem

Generalizes Multiway Cut and $k$-Cut

Given $G = (V, E)$, $w: E \rightarrow \mathbb{R}^+$
$T \subseteq V$: terminals
integer $k$, $k \leq |T|

Goal: find min-wt set of edges to remove such that $G$ is partitioned into $k$ components each of which contains at least one terminal from $T$
The Steiner k-Cut problem

Given $G = (V, E)$, $w: E \rightarrow \mathbb{R}^+$

$T \subseteq V$: terminals

integer $k$, $k \leq |T|$

**Goal:** find min-wt set of edges to remove such that $G$ is partitioned into $k$ components each of which contains at least one terminal from $T$

$k = |T|$ gives multiway cut problem

$T = V$ gives k-Cut problem
Greedy Splitting/Gomory-Hu tree algs

Greedy splitting naturally defined for the problem
Difficult to analyze directly but yields $2 - \frac{2}{k}$ approximation

Gomory-Hu tree based algorithm
Pick $e_1, e_2, ..., e_{k-1}$ from $T$ iteratively such that each new edge creates a new component with a terminal. Among possible edges choose one of min weight

Theorem: $2 - \frac{2}{k}$ approx for Steiner $k$-Cut
More general problem

Now instead of edges we allow arbitrary submodular functions

Given $V$ and a function $f : 2^V \rightarrow \mathcal{R}^+$

$T \subseteq V$

integer $k \leq |T|$

Goal: partition $V$ into $V_1, V_2, \ldots, V_k$ such that $V_i \cap T \neq \emptyset$ for $1 \leq i \leq k$ so as to minimize $\sum_i f(V_i)$
More general problem

Given $V$ and a function $f : 2^V \rightarrow \mathcal{R}^+$

$T \subseteq V$

integer $k \leq |T|$

Goal: partition $V$ into $V_1, V_2, \ldots, V_k$ such that $V_i \cap T \neq \emptyset$ for $1 \leq i \leq k$ so as to minimize $\sum_i f(V_i)$

Theorem: If $f$ is submodular and symmetric

Greedy splitting yields a $2\cdot 2/k$ approximation

(also for some other cases of $f$)
More general problem

**Theorem:** If $f$ is *submodular* and *symmetric*

Greedy splitting yields a $2-2/k$ approximation

(also for some other cases of $f$)

If $G$ is a graph or a hypergraph then the cut function $f(V_i) = w(\delta(V_i))$ is a symmetric submodular function

Hence Steiner $k$-cut is a special case
Why does it work?

Direct proof using submodularity [Zhao-Nagamochi-Ibaraki’05]

Another proof:
If $f$ is submodular and symmetric then it also admits a Gomory-Hu tree!

In other words, Gomory-Hu tree exists because cut-functions in graphs are submodular and symmetric
Greedy Splitting for Steiner k-Cut

We adapt proof of [Zhao-Nagamochi-Ibaraki’05] for submodular splitting problems and simplify it for the Steiner k-Cut problem.

Some notation
Greedy generations vertex partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k-1}$ starting with $\mathcal{P}_0 = \{V\}$

$\mathcal{P}_i$ is a refinement of $\mathcal{P}_{i-1}$ with one of the components of $\mathcal{P}_{i-1}$ split into two components.
Greedy Splitting for Steiner $k$-Cut

Some notation
Greedy generations vertex partitions $P_1, P_2, \ldots, P_{k-1}$ starting with $P_0 = \{V\}$

$P_i$ is a refinement of $P_{i-1}$ with one of the components of $P_{i-1}$ split into two components.

We are interested only in partitions that are valid: that is each component contains a terminal.
Analysis of Greedy Splitting

Let $w(P_i)$ the cost of edges cut in the partition $P_i$

We prove the following by induction on $i$

Lemma:
For any valid partition $P = \{V_1, V_2, ..., V_i\}$

\[ w(P_{i-1}) \leq w(\delta(V_1)) + w(\delta(V_2)) + ... + w(\delta(V_{i-1})) \]

(note that rhs doesn’t depend on $V_i$)
Analysis of Greedy Splitting

For any valid partition \( \mathcal{P} = \{V_1, V_2, \ldots, V_i\} \)
\[
w(\mathcal{P}_{i-1}) \leq w(\delta(V_1)) + w(\delta(V_2)) + \ldots + w(\delta(V_{i-1}))
\]

Suppose above is true: consider an optimum solution for problem \( \{A_1, A_2, \ldots, A_k\} \) where the ordering is chosen s.t
\[
w(\delta(A_k)) \geq w(\delta(A_i)) \text{ for } 1 \leq i \leq k-1
\]
Analysis of Greedy Splitting

Suppose above is true: consider an optimum solution for problem \( \{A_1, A_2, \ldots, A_k\} \) where the ordering is chosen s.t

\[
w(\delta(A_k)) \geq w(\delta(A_i)) \text{ for } 1 \leq i \leq k-1
\]

From the lemma with \( i = k-1 \),

\[
w(P_{k-1}) \leq (1-1/k) (w(\delta(A_1)) + \ldots + w(\delta(A_k))) \leq 2(1-1/k) \text{ OPT}
\]
Proof of the lemma

Base case $i = 1$ is easy to check

Assume hypothesis holds for $i=1$
Let $\mathcal{P} = (V_1, V_2, \ldots, V_i)$ be an arbitrary valid partition into $i$ components

Consider $\mathcal{P}_{i-2}$:
There must be some component $W \in \mathcal{P}_{i-2}$ and two indices $h < l$ with $W \cap V_h$ and $W \cap V_l$ both containing a terminal (Why?)
Proof of the lemma

Therefore splitting $W$ into $W \cap V_h$ and $W - V_h$

would be considered in the Greedy Splitting algorithm when refining $P_{i-2}$ into $P_{i-1}$

Since Greedy chose a cheapest split

$$w(P_{i-1}) - w(P_{i-2}) \leq \text{increase in cost if Greedy chose to split } W \text{ as above}$$

We claim that splitting $W$ as above increases cost by at most $w(\delta(V_h))$
Proof of the lemma

We claim that splitting $W$ as above increases cost by at most $w(\delta(V_h))$

Any new edge $e$ induced by the split of $W$ is from $(V_h \cap W)$ to $(W - V_h)$ so must be in $\delta(V_h)$
Proof of lemma

Therefore $w(P_{i-1}) - w(P_{i-2}) \leq w(\delta(V_h))$

Apply induction hypothesis on $i-1$ to the partition

$P' = \{V_1, \ldots, V_{h-1}, V_{h+1}, \ldots, V_{i-1}, V_h \cup V_i\}$

(we removed $V_h$ and merged it with $V_i$, use the fact that $h < l \leq i$)
Proof of lemma

Therefore \( w(P_{i-1}) - w(P_{i-2}) \leq w(\delta(V_h)) \)

Apply induction hypothesis on \( i-1 \) to the partition
\( P' = \{V_1, \ldots, V_{h-1}, V_{h+1}, \ldots, V_{i-1}, V_h \cup V_i\} \)

\[ w(P_{i-2}) \leq \left( \sum_{j=1}^{i-1} w(\delta(V_j)) \right) - w(\delta(V_h)) \quad \text{(since } V_h \text{ is no longer there in } P') \]

Combining above two inequalities proves the hypothesis for \( i \)
Exercises/Reading

Read about Gomory-Hu tree construction in Vazirani’s book, Chapter 4 exercise

Prove that the Greedy Split algorithm is optimal when G is a tree

Try to extend the analysis of the Gomory-Hu tree based algorithm for k-Cut to the algorithm for the Steiner k-Cut problem