APTAS for Bin Packing

Bin Packing has an asymptotic PTAS (APTAS)
[de la Vega and Leuker, 1980]

For every fixed $\varepsilon > 0$ algorithm outputs
a solution of size $(1+\varepsilon)OPT + 1$
in time polynomial in n

APTAS for Bin Packing

Split items into large and small
item i is large if $s_i \geq \varepsilon$, otherwise small

Pack large items

Pack small items using greedy on top of large items

Packing large items: shifting trick

Idea: change instance such that # of distinct sizes is constant. Can then solve problem using dynamic programming

L: large items

$OPT \geq s(L) = \sum_{i \in L} s_i \geq \varepsilon |L|$

Grouping large items

For any integer $1 \leq k \leq |L|/k$, can partition L into $L_1, L_2, \ldots, L_k$ such that

- for $1 < i \leq k$, items in L_i are all smaller than smallest item in L_{i+1}
- $|L_1| \leq |L_2| = |L_3| = \ldots |L_k|$

Sort items and pick L_k = largest $|L|/k$ items, L_{k-1} the next largest $|L|/k$ items and so on

Shifting

a_i : size of smallest item in L_i

For $i = 1$ to $k-1$ do
    if $j \in L_i$, set $s'_j = a_{i+1}$

Let $L' = L_1 \cup L_2 \cup \ldots \cup L_{k-1}$ with new sizes

Claim: L' can be packed in OPT number of bins

Claim: # of distinct sizes in L' is $k-1$

Packing large items

L' can be packed in OPT bins in $O(n^{2k})$ time using dynamic programming (can do better using other methods)

Pack L_k in $|L_k|$ bins - each item separately

# of bins used is $OPT + |L_k| \leq OPT + 2|L|/k$

Choose $k = 2/\varepsilon^2$ (assuming $|L| \geq k^2$)

Packing large items

L' can be packed in OPT bins in $O(n^{2k})$ time using dynamic programming

\# of bins used is $OPT + |L_k| \leq OPT + 2|L|/k$

Choose $k = 2/\varepsilon^2$

\# of bins used is $OPT + \varepsilon^2 |L|$

$\leq OPT + \varepsilon\, OPT$ (since $OPT \geq \varepsilon |L|$)

$\leq (1+\varepsilon)\, OPT$

running time is $O(n^{4/\varepsilon^2})$

Packing large items

Suppose $|L| \leq 4/\varepsilon^4$?

Lemma: Optimum solution for a Bin Packing problem can be computed in $O(n \log n\ 2^n)$ time

If $|L| \leq 4/\varepsilon^4$ can compute optimum solution in $2^{O(1/\varepsilon^4)}$ time

Packing small items

m: bins used to pack large items

Run greedy with small items

m': total # of bins used after small item packing

Claim: $m' \leq \max \{ m, \lceil (\sum_i s_i)/(1-\varepsilon) \rceil \}$

$\lceil (\sum_i s_i)/(1-\varepsilon) \rceil \leq (\sum_i s_i)(1+2\varepsilon) + 1$ for sufficiently small ε

Packing small items

m: bins used to pack large items

Run greedy with small items

m': total # of bins used after small item packing

Claim: $m' \leq \max \{ m, \lceil (\sum_i s_i)/(1-\varepsilon) \rceil \}$

Proof: if $m' > m$ then at most one bin $(1-\varepsilon)$-full

$(1-\varepsilon)(m'-1) < \sum_i s_i$

Packing small items

m: bins used to pack large items

Run greedy with small items

m': total # of bins used after small item packing

Claim: $m' \leq \max \{ m, \lceil (\sum_i s_i)/(1-\varepsilon) \rceil \}$

$m \leq (1+\varepsilon)$ OPT

therefore $m' \leq (1+2\varepsilon)$ OPT + 1 for sufficiently small ε ($\leq$ ½)

Summary of APTAS

$m' \leq (1+2\varepsilon)\ \text{OPT} + 1$

Running time: dominated for packing large items

$n^{O(1/\varepsilon^2)} + 2^{O(1/\varepsilon^4)}$

Better algorithms for Bin Packing

[Karmarkar-Karp, 1982] : sophisticated ideas

Polytime algorithm that to pack using
$OPT + \log^2 OPT$ bins

Also AFPAS (asymptotic FPTAS): $(1+\varepsilon)\ OPT + f(\varepsilon)$ bins in time $poly(n, 1/\varepsilon)$

Open problem: is there a poly-time algorithm to pack using $OPT + 1$ bins? $OPT + c$ bins where c is some absolute constant? Sub-exponential algorithm?

Multiprocessor scheduling with precedence constraints

Jobs J_1, J_2, ..., J_n to be executed on m identical processors/machine M_1, M_2, ..., M_m

Job J_i has processing time/size p_i

Jobs have *precedence constraints* between them
$J_i \prec J_k$ implies J_i cannot be done before J_k completes

Directed acyclic graph DAG G encodes constraints

Precedence constraints

Encode dependencies which prevent parallelism

Data and control dependencies

Two improtant applications

- Parallel programming
- Instruction scheduling in multi-issue processors

Scheduling Problem

Jobs to be assigned to machines

Non-preemptive schedule: once job is started cannot interrupt it

C_j : finish time of job j

$s_j = C_j - p_j$: start time of job j

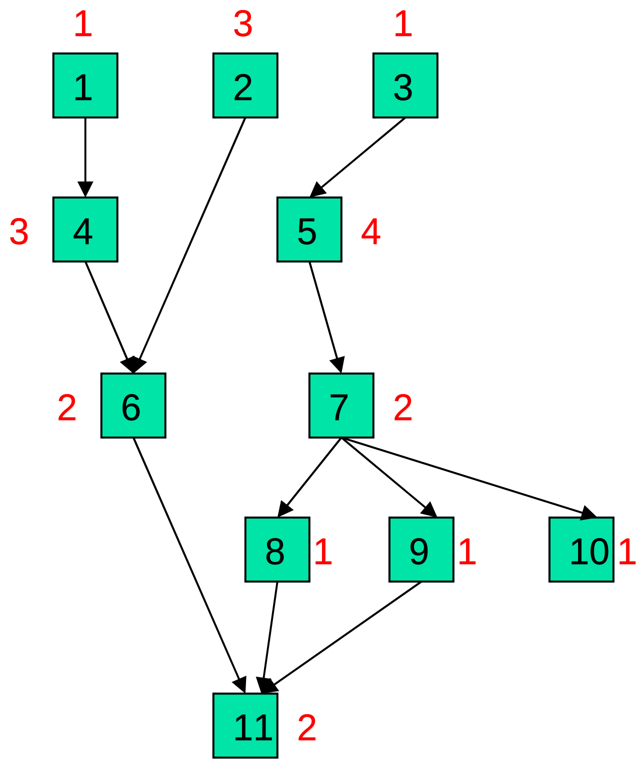
j occupies *time slots* $s_j+1, ..., C\_j$

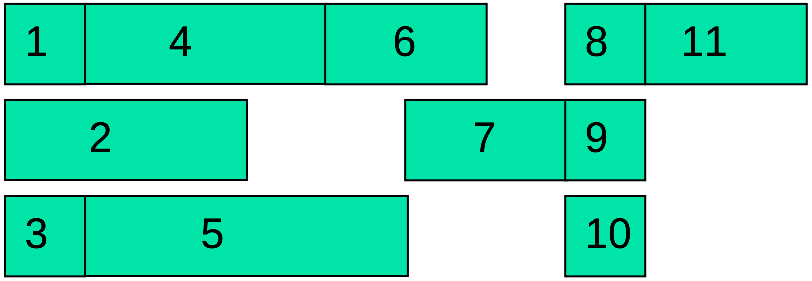
minimize $\max_j C_j$

Example

$m = 3$

$C_{max} = 10$





Processing times: red

List Scheduling

List: ordering of jobs according to some priority

Higher priority to jobs earlier in the list

Assume that list is a topological sort of the dag

Greedy list scheduling algorithm: schedule as early as possible

Job i is *ready* at t if all jobs in $Pred(i)$ completed by t

List Scheduling

```
for k = 1 to m do
   a_k = 0        // a_k time when M_k is available
for t = 0 to T_max do
  if no jobs left to schedule break
  for k = 1 to m do
     if a_k > t  continue
     if no ready jobs break
     i ← highest priority ready job
     a_k ← t + p_i
     remove i from list
```

Predecessors and Successors

Take transitive closure or DAG

$Pred(i) = \{ j \mid j \prec i \}$ predecessors of i

$Succ(i) = \{ j \mid i \prec j \}$ successors of i

Chain: $i_1 \prec i_2 \prec \ldots \prec i_k$

List Scheduling Analysis

Theorem: List scheduling with any list is a 2-1/m approximation

Analysis:

Two lower bounds:

$OPT \geq \sum_i p_i/m$ (average load) (LB1)

$OPT \geq \sum_{i \in A} p_i$ for any *chain* A

$OPT \geq \max_{A:\ A\ \text{chain}} p(A)$ (LB2)

List Scheduling Analysis

Two lower bounds:

$OPT \geq \sum_i p_i/m$ (average load)

$OPT \geq \sum_{i \in A} p_i$ for any *chain* A

$C_{max} = \max_j C_j$ maximum completion time in list schedule

Theorem: $C_{max} \leq LB1 + LB2$

List Scheduling Analysis

Theorem: $C_{max} \leq LB1 + LB2$

$t \in (0, C_{max}]$

t is a *full time slot* if all machines busy in t

otherwise t is a *partial time slot*

# of full time slots $\leq \sum_i p_i/m = LB1$

To prove: # of partial time slots $\leq p(A)$ for some chain A

List Scheduling Analysis

Theorem: $C_{max} \leq LB1 + LB2$

# of full time slots $\leq \sum_i p_i/m = LB1$

To prove: # of partial time slots $\leq p(A)$ for some chain A

C_{max} = # of full time slots + # of partial slots

$\leq LB1 + p(A) \leq LB1 + LB2$

# of full time slots $\leq (\sum_i p_i - p(A))/m$

hence $C_{max} \leq LB1 + (1\text{-}1/m)\ LB2 \leq (2\text{-}1/m)\ OPT$

List scheduling analysis

Create chain inductively

i_1 : job such that $C_{i_1} = C_{max}$ (last job to complete)

let $t_1 = C_{i_1}$

t_2: max integer in $[0, s_{i_1}]$ s.t t_2 is a partial slot

if $t_2 = 0$ stop: chain is only i_1

Claim: some predecessor of i_1 is executing at t_2, otherwise i_1 would have been scheduled at t_2

List scheduling analysis

i_1 : job such that $C_{i_1} = C_{max}$ (last job to complete)

let $t_1 = C_{i_1}$

t_2: max integer in $[0, s_{i_1}]$ s.t t_2 is a partial slot

if $t_2 = 0$ stop: chain is only i_1

i_2: (any) predecessor of i_1 executing at t_2

$t_3 =$ max integer in $[0, s_{i_2}]$ s.t t_3 is a partial slot

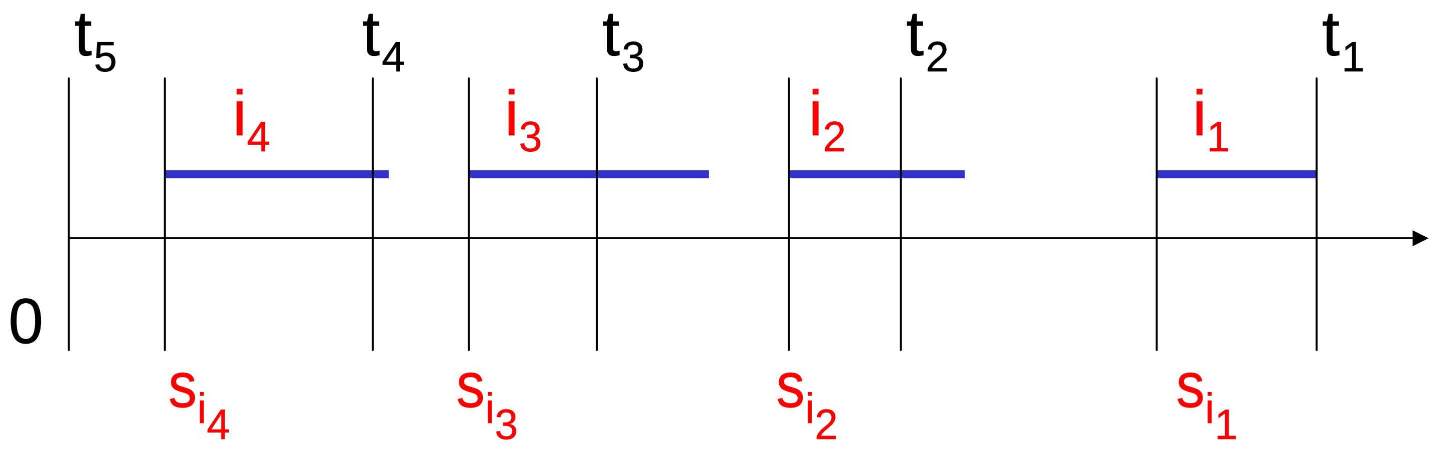
List scheduling analysis

inductively

t_k: max integer in $[0, s_{i_k}]$ s.t t_k is a partial slot
stop if $t_k = 0$
else
i_{k+1} : predecessor of i_k that is executing at t_k

Let $i_1 \succ i_2 \succ \ldots \succ i_h$ be the chain created
$t_1 > t_2 > \ldots > t_k \geq t_{k+1} = 0$ be associated times

Picture



Note: execution intervals of jobs in chain don’t overlap

List scheduling analysis

Let $i_1 \succ i_2 \succ \ldots \succ i_h$ be the chain created

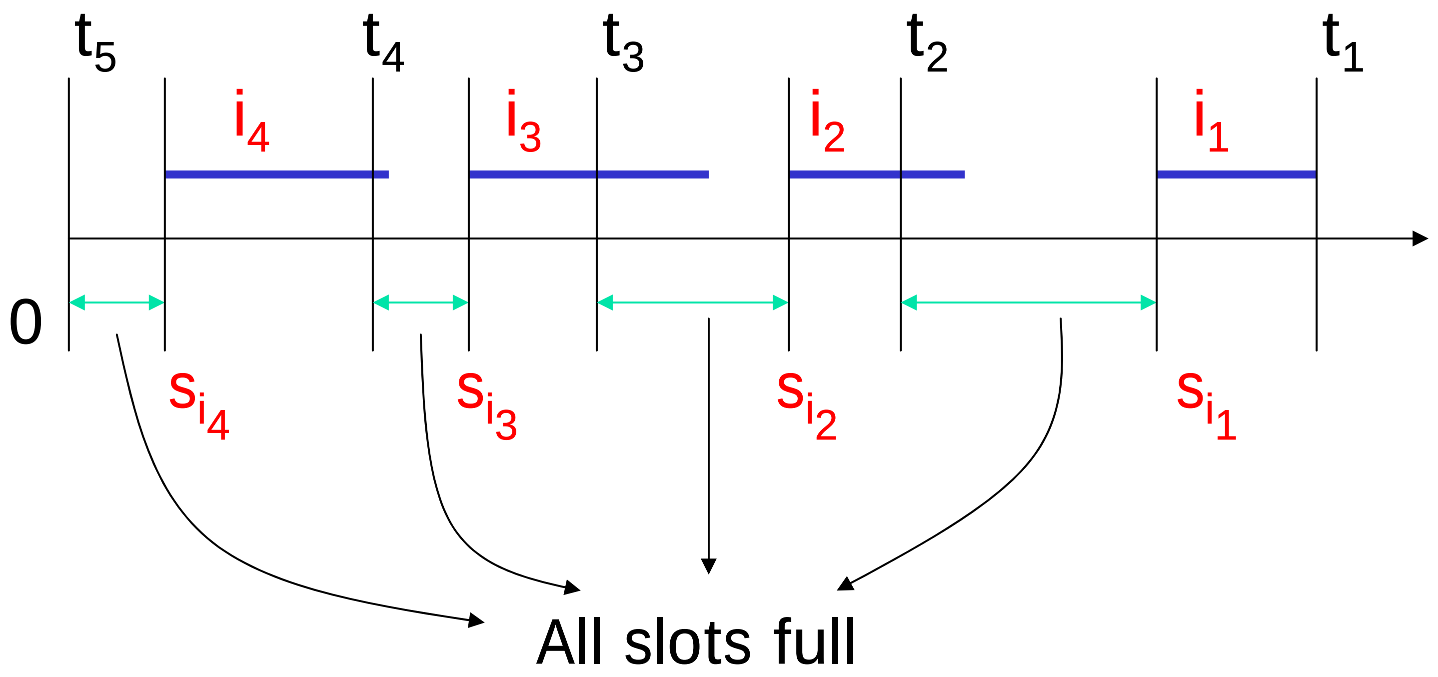
$t_1 > t_2 > \ldots > t_k \geq t_{k+1} = 0$ be associated times

Claim: for $1 \leq j \leq k$

1. all slots in $[t_{j+1}+1, s_{i_j}]$ are full
2. i_j executing in $[s_{i_j}+1, t_j]$

All slots that are partial can be charged to some interval in which a job from chain is executing

Picture



Critical Path Scheduling

Create a good list: give importance to jobs that are “critical” for other jobs

i_1: job with longest chain (break ties arbitrarily)

Remove i_1 from DAG

i_2: job with longest chain

Remove i_2 from DAG

…

Critical Path Scheduling

Works very well in practice

Several rules to break ties

However there are examples which show that critical path scheduling does not improve 2-approx (irrespective of how you break ties)

More on scheduling

Open problem: obtain 2-ε approx ($\geq$ 35 years)

even for the case when jobs are all unit size

For unit job sizes

Theorem: Unless P=NP no approximation ratio better than 4/3-ε for any $\varepsilon > 0$

Proof: reduction from *clique*

Reduction from clique

Q: Given $G=(V,E)$ and integer k, does G have a clique of size k?

NP-Complete

Given $G=(V,E)$, k create a scheduling instance as follows

Two classes of jobs: graph jobs, dummy jobs

m machines where $m = n(n-1)/2 + 1$

Reduction from clique

Two classes of jobs: graph jobs, dummy jobs

$|E| + |V|$ graph jobs, one for each edge and one for each vertex

If J_e is a job corresponding to edge $e = uv$

then J_u and J_v are predecessors for J_e

Dummy Jobs: three sets of jobs, D_1, D_2, D_3

$|D_1| = m - k$

$|D_2| = m - k(k-1)/2$

$|D_3| = m - |E| + k(k-1)/2$

Reduction from clique

Dummy Jobs: three sets of jobs, D_1, D_2, D_3

$|D_1| = m – k$

$|D_2| = m – k(k-1)/2 - |V| + k$

$|D_3| = m - |E| + k(k-1)/2$

Each job in D_1 is a predecessor to each job in D_2

Each job in D_2 is a predecessor to each job in D_3

Clearly $OPT \geq 3$ from dummy jobs

Reduction from clique

Claim: If G has a clique of size k then $OPT = 3$

Dummy jobs have to be scheduled in order

All D_1 in slot 1, all D_2 in slot 2, all D_3 in slot 3

Schedule clique vertices in slot 1

Schedule edges of clique in slot 2 + schedule remaining vertices in slot 2

Schedule remaining edges in slot 3

Reduction from clique

Claim: If OPT = 3 then G has a clique of size k

Essentially the same as previous claim.

Therefore if G does not have a clique of size k then OPT $\geq$ 4

Implies that 4/3 -ε approx not possible unless P=NP

Open Problem

For unit jobs and m *fixed* (independent of # of jobs) it is unknown whether the problem is NP-Complete even for m=3!

Open for more than 30 years

Obtain 2-ε approximation for m fixed