Multicut

Given undirected graph $G=(V,E)$, edge costs, $c: E \to \mathcal{R}^+$

Pairs of nodes, s_1t_1, s_2t_2, ..., s_kt_k

Goal: remove min-cost set of edges $E' \subset E$ such that for $1 \leq i \leq k$, s_i and t_i are separated

Generalizes multiway-cut and hence is APX-hard to approximate

LP Relaxation

As before we consider a simple metric relaxation.

$l_e \in \{0, 1\}$ to indicate if e is cut or not

in LP relaxation we use $l_e \in [0,1]$ for length of e

P_i : set of all paths from s_i to t_i

$\min \sum_e c_e l_e$

s.t

$\sum_{e \in p} l_e \geq 1 \qquad p \in P_i, 1 \leq i \leq k$

$l \geq 0$

LP Relaxation

The dual of the LP is the following:
let $P = \cup_i P_i$

$f(p)$: variable for $p \in P$, (flow on path p)

$\max \sum_{p \in P} f(p)$

s.t
$\sum_{p:\, e \in p} f(p) \leq c_e$ for all $e \in E$
$f(p) \geq 0$

Thus dual is a multicommodity flow problem that maximizes the total flow for all pairs

Flow-cut gap

Both primal (the cut formulation) and the dual (the flow formulation) can be written in compact form with a polynomial number of constraints (how?).

Note that when k=1, that is the single pair case, we have the maxflow-mincut theorem which states that the total flow is equal to the minimum cut.

However for arbitrary k the multicut can be much larger than the total flow

Observe that the integrality gap of the LP for multicut is precisely the gap between the flow and the cut!

Integrality gap of the LP

Theorem: The integrality gap of the given LP for multicut is $O(\log k)$.

Therefore the above theorem shows that the total flow is at least $\Omega(1/\log k)$ for the minimum cut

Theorem: The integrality gap of the given LP for multicut is $\Omega(\log k)$.

The above theorem shows that there are examples where the flow is $O(1/\log k)$ of the minimum cut.

Rounding the LP

There are several different ways to round the LP

Vazirani's book gives the original proof based on region growing

Here we give a proof based on a more recent randomized rounding procedure

Let $B_l(v, r)$ denote the ball of radius r around node v in the metric induced by the lenghts l given by the LP

Rounding Algorithm

Pick a number θ uniformly at random from $[0, ½)$

Pick a random permutation π of $\{1,2,\ldots,k\}$

for each node v, let $color(v) = 0$ // no color assigned

For $i = 1$ to k do

  for each u in $B_l(s_i, \theta)$

    if ($color(u) = 0$) // still unassigned

      set $color(u) = i$ // assigned only once

end for

$E' = \{ uv \mid color(u) \neq color(v) \}$

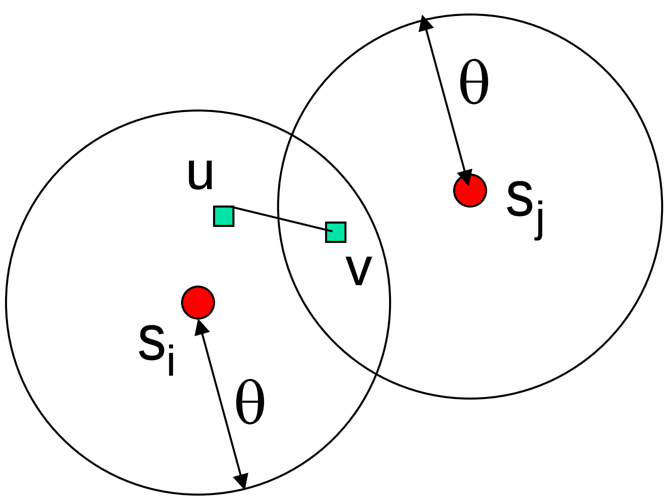
Output E' as the cut

Rounding algorithm

Let $V_i = \{ u \mid color(u) = i \}$

Note that $E' = \cup_{i=1}^{k} \delta(V_i)$

In picture the edge uv is not cut if $\pi(i) < \pi(j)$ but might be cut otherwise



Analysis

Claim: E' is a feasible multicut

To see this, we claim that s_i and t_i do not *both* belong to any V_j

Suppose they did. Then since $V_j \subseteq B_l(s_j,\theta)$,
both s_i, $t_i \in B_l(s_j,\theta)$ which is a contradiction since that would imply that $l(s_i t_i) \leq 2\theta < 1$

Analysis

Claim: $\Pr[e=uv \text{ is cut}] \leq 2\, H_k\, l_e$ where H_k is the k'th harmonic number

Therefore $\text{Expect}[c(E')] \leq 2\, H_k\, OPT_{LP}$

To prove the claim we consider the event A_i that $e=uv$ is *cut first by* s_i

More precisely, the event that exactly one of $\{u, v\}$ gets color i *and* $\min\{\text{color}(u), \text{color}(v)\} \in \{0, i\}$

Clearly $\Pr[e=uv \text{ is cut}] = \sum_{i=1}^{k} \Pr[A_i]$

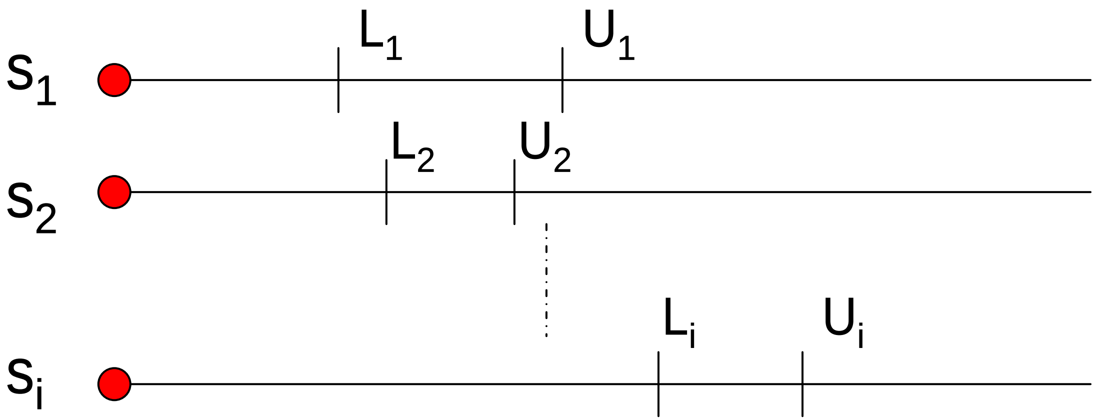
Analysis

We focus on $e=uv$

Let $L_i = \min(l(s_i,u), l(s_i,v))$

and $U_i = \max(l(s_i,u), l(s_i,v))$

Wlog assume that terminals are renumbered such that $L_1 \leq L_2 \leq \ldots \leq L_k$



nodes ordered in *increasing* l-distance from terminals

Analysis

Fix some value r for θ

What is $\Pr[A_i \mid \theta = r]$?

A_i happens only if $r \in [L_i, U_i)$ *and* s_i comes before $s_1, s_2, \ldots, s_{i-1}$ in the random permutation π

Therefore

$\Pr[A_i \mid \theta = r] \leq 1/i$ for all r

Since π and θ are chosen independently

$\Pr[A_i] \leq (1/i) \Pr[r \in [L_i, U_i)] \leq 2 l_e / i$

since $U_i - L_i \leq l_e$

Analysis

Therefore $\Pr[e=uv \text{ is cut}] = \sum_{i=1}^{k} \Pr[A_i] \leq 2H_k\, l_e$

Lower bound on integrality gap

We now show instances in which the LP integrality gap is $\Omega(\log k)$

We use constant degree expanders for this

A graph $G=(V,E)$ is an α-edge-expander for some real valued α if forall $S \subset V$, $|S| \leq |V|/2$

$|\delta(S)| \geq \alpha\ |S|$

If $\alpha = 1$ we call G an expander

Lower bound on integrality gap

The following can be shown by the probabilistic method

Theorem: For any $k > 2$, there exist infinitely many n-node graphs G, such that G is k-regular and G is an expander

Explicit constructions of expanders are also available.

For our lower bound we let G be a 3-regular expander (any constant degree would do and any constant α would do as well)

Let $d_G(uv)$ be the distance between u and v in the graph G (with edge lengths $= 1$)

Lower bound on integrality gap

Let diam(G) be the diameter of G

Since G is of degree 3, we can easily see that $diam(G) \geq \log_3 n$ (Why?)

Since G is an expander, it can be seen that $diam(G) \leq c \log_3 n$ for some constant c (Exercise)

Let $P = \{ uv \in V \times V \mid d_G(uv) \geq ½ \log_3 n \}$

We create a multicut instance from G by asking all pairs in P to be separated.

We set $c_e = 1$ for $e \in E$

Lower bound on integrality gap

We obtain a feasible solution to the LP on the given instance as follows:

we set $l_e = 2/\log_3 n$ for each edge $e \in E$

It is easy to see that for all uv, $l(uv) = d_G(uv)\ 2/\log_3 n$

Therefore for all $uv \in P$, $l(uv) \geq 1$

Hence $OPT_{LP} \leq \sum_e c_e l_e = \sum_e l_e = 2m/\log_3 n$

where m is # of edges

Since G is 3-regular, $m = 3n/2$ and hence
$OPT_{LP} \leq 3n/\log_3 n$

Lower bound on integrality gap

Let E' be any feasible (integral) solution and let $V_1, V_2, \ldots, V_h$ be the connected components in the graph $G[E \setminus E']$

Claim: for $1 \le i \le h$, $|V_i| \le n/2$

Assuming claim,

$|E'| = ½ \sum_{i=1}^{h} |\delta(V_i)|$

but since G is an expander and $|V_i| \le n/2$ for each i

$|\delta(V_i)| \ge |V_i|/2$ for each i and hence

$|E'| \ge n/2$

Therefore $OPT \ge n/2$

Since $OPT_{LP} \le 3n/\log_3 n$, gap is $\Omega(\log k)$ (note $k = \Theta(n^2)$)

Lower bound on integrality gap

Let E' be any feasible (integral) solution and let $V_1, V_2, \ldots, V_h$ be the connected components in the graph $G[E \setminus E']$

Claim: for $1 \le i \le h$, $|V_i| \le n/2$

for $u \in V$, let $P'_u = \{ v \mid uv \notin P \}$

Note that if v in P'_u then $d_G(uv) < \frac{1}{2} \log_3 n$

but since G is 3-regular $|P'_u| \le 3^{\frac{1}{2} \log_3 n} \le$ \sqrt{n}

For any V_i, and a node $u \in V_i$, we have $|V_i| \le P'_u$ for

otherwise there is some node $v \in V_i$ with $uv \in P$

This shows that $|V_i| \le$ \sqrt{n} $\le n/2$