k-Median problem

F : facilities

D : clients

integer k

Feasible solution: k facilities S from F, $\sigma: D \rightarrow S$

Goal: minimize $\text{cost}(S, \sigma) = \sum_{j \in D} c_{\sigma(j), j}$

Local Search for k-Median

Start with S_0 any set of k facilities

Local search move: *swap a facility*

$S' = S - l + i$ where $l \in S$ and $i \notin S$

In each iteration pick best swap
and STOP if no improvement possible

(can use approximate stopping condition)

Local Search for k-Median

Theorem: Local optima is 5-approximate

Improved local search: swap p facilities at a time
$S' = S - A + B$ where $|A| = |B| = p$ and $A \subset S$,
$B \cap S = \emptyset$

Running time depends on n^p

Theorem: Local optima is $(3+2/p)$ approximate

Analysis for 5-approximation

Clever and perhaps not so intuitive argument

Let S be a local optimum and O be an optimum solution

implies for every $l \in S$ and $i \in O$

$cost(S-l+i) \geq cost(S)$

(we ignore the approximate stopping condition in this analysis)

Analysis

Set up k swap pairs carefully chosen

each facility $i \in O$ is in exactly *one* of the pairs

each facility $l \in S$ is at most *two* of the pairs

Let σ and ω be the assignment functions of clients to facilities for S and O

Analysis

A facility $l \in S$ is said to *capture* a facility $i \in O$ iff at least half the clients of i are clients of l

in other words $| \omega^{-1}(i) \cap \sigma^{-1}(l) | \geq | \omega^{-1}(i) |/2$

Intuitively l is close to i

Note: a facility in O is captured by at most one facility in S but a facility in S can capture multiple facilities in O

Analysis

Set up a bipartite graph H=(S,O) as follows

There is an edge (l, i) in H iff l captures i

Now we define the k swap pairs

- Remove all facilities in S with degree > 1 in H
- if (l, i) is an edge in H and degree of l is 1 then add (l, i) to the swap pair and remove l, i
- pair each remaining facility i' in O with some arbitrary remaining facility in S s.t no facility in S is paired more than twice *(check that this is possible by simple counting)*

Analysis

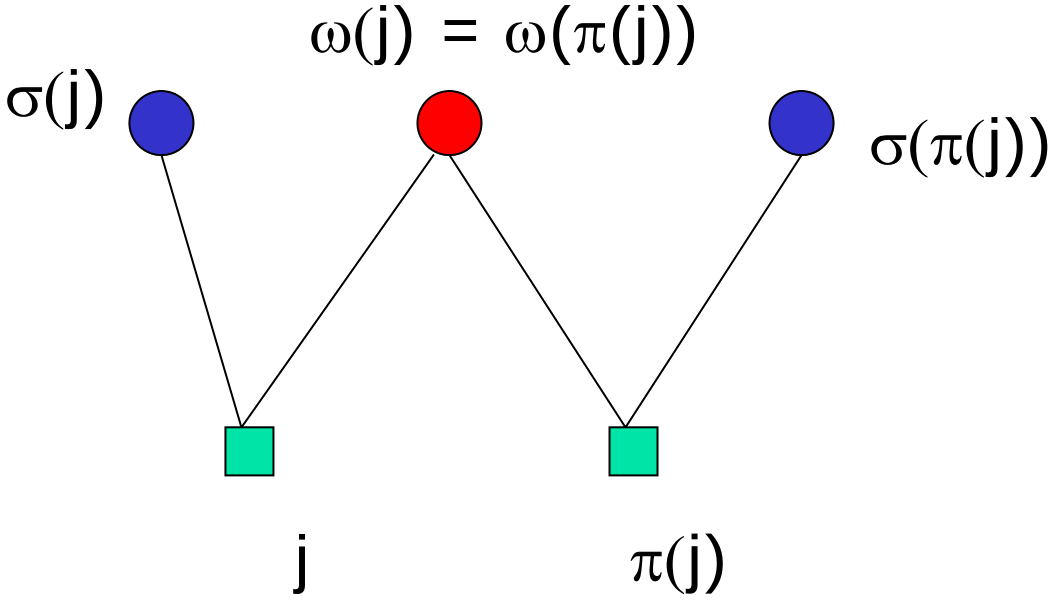
The final ingredient of the set up is a permutation π on the clients that satisfies some important properties

$\pi: D \rightarrow D$ permutation implies 1-1 onto function

Properties:

- j and $\pi(j)$ are assigned to the *same* facility in O, that is $\omega(j) = \omega(\pi(j))$
- j and $\pi(j)$ are assigned to *different* facilities in S unless $\omega(j)$ is captured in which case we don't care

Pic



Analysis

Can show that π satisfying properties exists

Consider $i \in O$ s.t i is not captured. Note that we can restrict attention to clients in $\omega^{-1}(i)$ and define π on them

For each $j \in \omega^{-1}(i)$ we need to ensure that $\pi(j)$ has the property that $\sigma(j) \neq \sigma(\pi(j))$

Analysis

- For each $j \in \omega^{-1}(i)$ we need to ensure that $\pi(j)$ has the property that $\sigma(j) \neq \sigma(\pi(j))$

- Wlog assume clients in $\omega^{-1}(i)$ are numbered $0,1,2,\ldots,h-1$ where $h = | \omega^{-1}(i) |$ and further that clients with same σ are consecutively numbered

- set $\pi(j) = j + \lceil h/2 \rceil \textit{ mod } h$

- Check that this satisfies the desired property (here is where the fact that i is not captured is used)

Analysis

Now we are ready to prove the 5-approximation

foreach swap pair (l, i),
cost(S – l + i) – cost(S) $\geq$ 0

lower bound $\sum_{(l,\ i)}$ (cost(S – l + i) – cost(S))
by
5 cost(O) – cost(S)
as follows

Analysis

Consider client j

how much does it contribute to

$\sum_{(l,\, i)} \ (\text{cost}(S - l + i) - \text{cost}(S)\ )$

we consider i when $i = \omega(j)$ and

when $(l=\sigma(j),\, i)$ is a swap pair (at most two pairs)

(in this second we assume $i \neq w(j)$)

for other values of i we can ignore j's contribution since it only helps the inequality (why?)

Analysis

when $i = \omega(j)$

j contributes

$c_{\omega(j),j} - c_{\sigma(j),j}$

when (l=\sigma(j), i) is a swap-pair j contributes ??

note that $i \neq \omega(j)$ implies $\omega(j)$ not captured by choice of swap pairs (why? think carefully here)

hence we can use $\pi(j)$ here

j is assigned to $\sigma(\pi(j))$ since $l=\sigma(j)$ is swapped out. here we use the fact that $\sigma(\pi(j))$ is different from $\sigma(j)$

Analysis

j is assigned to $\sigma(\pi(j))$ since $l=\sigma(j)$ is swapped out. here we use the fact that $\sigma(\pi(j))$ is different from $\sigma(j)$

let $r = \sigma(\pi(j))$

so j contributes $c_{r,j} - c_{\sigma(j)j}$

observe that this contribution is non-negative since $\sigma(j)$ is closer to j than r by defn

Analysis

so j contributes $c_{r,j} - c_{\sigma(j)j}$

observe that this contribution is positive since $\sigma(j)$ is closer to j than r by defn

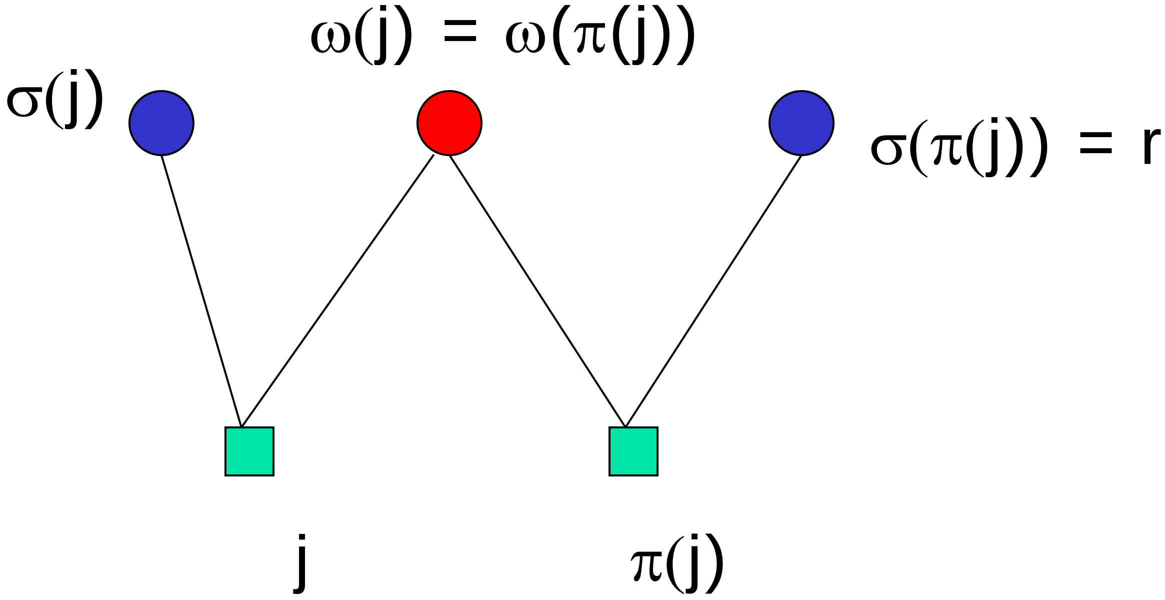
by triangle ineq

$$c_{r,j} \leq c_{\omega(j),j} + c_{\omega(j),\pi(j)} + c_{\pi(j),r}$$

note that there can be two (l, i) pairs so total contribution of j is at most

$$c_{\omega(j),j} - c_{\sigma(j),j} + 2(c_{\omega(j),j} + c_{\omega(j),\pi(j)} + c_{\pi(j),r} - c_{\sigma(j)j})$$

Pic



Analysis

- note that there can be two (l, i) pairs so total contribution of j is at most

$c_{\omega(j),j} - c_{\sigma(j),j} + 2(c_{\omega(j),j} + c_{\omega(j),\pi(j)} + c_{\pi(j),r} - c_{\sigma(j)j})$

- When we sum up over all j we notice that the last two terms disapper from the fact the π is a permutation. And also from the fact that $\omega(\pi(j)) = \omega(j)$ we get the total sum as

cost(O) – cost(S) + 4 cost(O)

Analysis

Therefore

cost(O) – cost(S) + 4 cost(O) \geq 0

which implies that

cost(S) $\leq$ 5 cost(O) = 5 OPT