A note on the Survivable Network Design Problem

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Survivable Network Design Problem (SNDP)

Input:
- undirected graph $G=(V,E)$
- integer requirement $r(st)$ for each pair of nodes $st$

Goal: min-cost subgraph $H$ of $G$ s.t $H$ contains $r(st)$ disjoint paths for each pair $st$
\[ r(s_1 t_1) = 2 \]
\[ r(s_2 t_2) = 2 \]
\[ r(s_3 t_3) = 1 \]
$r(s_1t_1) = 2$
$r(s_2t_2) = 2$
$r(s_3t_3) = 1$
SNDP Variants

Requirement

- EC-SNDP: paths are required to be edge-disjoint
- Elem-SNDP: element disjoint
- VC-SNDP: vertex/node disjoint

Cost

- edge-weights – focus of this talk
- node-weights
Known Approximations

<table>
<thead>
<tr>
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<th>Edge Weights</th>
<th>Node Weights</th>
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<tbody>
<tr>
<td>EC-SNDP</td>
<td>2 [Jain’98]</td>
<td>O(k log n) [Nutov’07]</td>
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<tr>
<td>Elem-SNDP</td>
<td>2 [FJW’01]</td>
<td>O(k log n) [Nutov’09]</td>
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<tr>
<td>VC-SNDP</td>
<td>O(k^3 log n) [CK’09]</td>
<td>O(k^4 log^2 n) [CK’09+Nutov’09]</td>
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k := max\_st r(st)
\[ \min \sum_e c(e) x(e) \]

\[ x(\delta(A)) \geq r(st) \quad A \subseteq V, \text{A separates st} \]

\[ 0 \leq x(e) \leq 1 \]

Cut-LP for EC-SNDP

\[ r(s_1t_1) = r(s_2t_2) = 2 \quad \text{and} \quad r(s_3t_3) = 1 \]
\[ \min \sum_{e} c(e) x(e) \]

\[ x(\delta(A)) \geq r(st) \quad A \subset V, \ A \text{ separates } st \]

\[ 0 \leq x(e) \leq 1 \]

**Theorem:** [Jain] Integrality gap of Cut-LP is 2
Jain’s iterated rounding

- Solve Cut-LP and find a *basic feasible solution* \( x \)

- There is always an \( e \) such that \( x(e) = 0 \) or \( x(e) \geq \frac{1}{2} \)
  - \( x \) defined by a *laminar family of tight sets*
  - *Counting argument* to obtain contradiction if no edge \( e \) such that \( x(e) = 0 \) or \( x(e) = 1 \) or \( x(e) \geq \frac{1}{2} \)

- Round up \( e \) with large value and recurse/iterate
  - *Residual problem* falls into more general class – covering skew supermodular requirement by a graph
Skew-supermodular functions

\[ f : 2^V \rightarrow \mathbb{Z} \text{ is skew-supermodular if for all } A, B \subseteq V \]

\[ f(A) + f(B) \leq f(A \cup B) + f(A \cap B) \]

or

\[ f(A) + f(B) \leq f(A - B) + f(B - A) \]

Requirement function of SNDP is skew-supermodular
Covering skew-supermodular requirement by graph

Given $G=(V,E)$ and skew-supermodular $f : 2^V \rightarrow \mathbb{Z}$

Find $E' \subseteq E$ of min cost such that for all $A \subseteq V$

$$|\delta_{E'}(A)| \geq f(S)$$

Key point: Given $G$, $f$ and subset $E' \subseteq E$ the residual function $f'$ defined by $f'(A) = f(A) - |\delta_{E'}(A)|$ is skew-supermodular
Covering skew-supermodular requirement by graph

**Key point:** Given $G$, $f$ and subset $E' \subseteq E$ the *residual function* $f'$ defined by $f'(A) = f(A) - |\delta_{E'}(A)|$ is skew-supermodular

$|\delta_{E'}|$ is a *symmetric submodular function*

$g(A) + g(B) \geq g(A \cup B) + g(A \cap B)$

and

$g(A) + g(B) \geq g(A - B) + g(B - A)$
Residual problem

\( f : 2^V \rightarrow \mathcal{Z} \) is skew-supermodular if for all \( A, B \subseteq V \)

\[
\begin{align*}
    f(A) + f(B) & \leq f(A \cup B) + f(A \cap B) \quad \text{or} \\
    f(A) + f(B) & \leq f(A - B) + f(B - A)
\end{align*}
\]

\( g : 2^V \rightarrow \mathcal{Z} \) is a symmetric submodular function

\[
\begin{align*}
    g(A) + g(B) & \geq g(A \cup B) + g(A \cap B) \quad \text{and} \\
    g(A) + g(B) & \geq g(A - B) + g(B - A)
\end{align*}
\]

Implies \( f - g \) is skew supermodular
Contributions of paper

- Counting argument delicate – several different proofs over the years
  - This paper: another proof which is more “combinatorial” based on a new inductive claim

- Reduction of element connectivity to edge connectivity via hypergraphs
  - Based on paper of [Zhao-Nagamochi-Ibaraki’01]
  - Eliminates complex notation of set pairs etc
  - Easy to teach or give as an exercise
  - Connections to node weights – see paper
Element-SNDP

- Interpolates EC-SNDP and VC-SNDP
- Independent interest, other connections etc
- $G=(V,E)$ and $V$ partitioned into terminals $T$ and non-terminals $N$
  - Requirement *only* between terminals
  - Paths for a terminal pair should be disjoint in *elements* (edges and non-terminals)
  - Wlog assume $T$ is an independent set: then paths should be disjoint in non-terminals
\[ r(s_1t_1) = 2 \]
\[ r(s_2t_2) = 3 \]
\[ r(s_3t_3) = 1 \]
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2-approx for Elem-SNDP

- Based on iterated rounding [Fleischer-Jain-Williamson, Cheriyan-Vetta-Vempala]
- Needs a set-pair based relaxation and generalization of Jain’s argument
Hypergraphic SNDP

[Zhao-Nagamochi-Ibarakai]

\[ G=(V,E) \] a hypergraph, each edge \( e \) in \( E \) a subset of \( V \)

Integer requirements \( r(st) \) for each pair \( st \) of vertices

Find min cost subset of hyperedges such that for each cut \( \delta(A) \) that separates \( st \) need \( r(st) \) hyperedges cross \( A \)

Generalizes EC-SNDP to Hypergraphs
Residual problem

\[ f : 2^V \to \mathbb{Z} \text{ is skew-supermodular} \]
\[ g : 2^V \to \mathbb{Z} \text{ is a symmetric submodular function} \]

Implies \( f - g \) is skew supermodular

Hypergraph cut function is symmetric submodular

Main issue for iterated rounding: counting argument
Reduction of Elem-SNDP to Hypergraphic SNDP
Reduction of Elem-SNDP to Hypergraphic SNDP
After reduction

• New hyperedges have zero cost

• Include all hyperedges in solution

• Residual problem is covering skew supermodular function by edges

• 2-approximation for residual problem follows from Jain’s argument for EC-SNDP
Jain’s iterated rounding

• Solve Cut-LP and find a basic feasible solution $x$

• There is always an $e$ such that $x(e) = 0$ or $x(e) \geq \frac{1}{2}$
  • $x$ defined by a laminar family of tight sets
  • Counting argument to obtain contradiction if no edge $e$ such that $x(e) = 0$ or $x(e) = 1$ or $x(e) \geq \frac{1}{2}$

• Round up $e$ with large value and recurse/iterate
  • Residual problem falls into more general class – covering skew supermodular requirement by a graph
x is basic feasible solution and $0 < x(e) < 1$ for all e
Set $S$ tight if $x(\delta(S)) = f(S)$

**Theorem [Jain]** x unique solution to system of m tight sets that form a *laminar* family
Counting argument: easy case

\[ x \text{ is basic feasible solution and } 0 < x(e) < 1 \text{ for all } e \]
\[ x \text{ unique solution to system of } m \text{ tight laminar sets } \]
Claim: $f(S) \geq \alpha(S) - \beta(S)$

$\alpha(S)$: number of descendants of $S$

$\beta(S)$: number of edges with both end points inside $S$
Open Problem(s)

Can ideas help with *improvements/simplifications* for *degree bounded* network design? Especially element connectivity.
Thank You!