Structure of Large-Treewidth Graphs: Recent Developments

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Joint work with
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Tree decompositions and Treewidth

Studied by [Halin’76]

Again by [Robertson & Seymour’84] as part of their graph minor project
Tree decompositions and Treewidth

Fundamental graph parameter

- key to graph minor theory
- generalizations to matroids via branchwidth
- algorithmic applications
- connections to theoretical computer science
Recent Progress on Approximation Algorithms for Disjoint Paths

[Chuzhoy’11] “Routing in Undirected Graphs with Constant Congestion”

[Chuzhoy-Li’12] “A Polylogarithmic Approximation for Edge-Disjoint Paths with Congestion 2”

[C-Ene’13] “Polylogarithmic Approximation for Maximum Node Disjoint Paths with Constant Congestion”

Answering a graph theoretic conjecture of [C-Khanna-Shepherd]
Some recent developments in treewidth

[C-Chuzhoy’12] “Large-treewidth graph decompositions and applications”

[C-Chuzhoy’13] “Polynomial bounds for the Grid-Minor Theorem”

Ongoing work
Tree width and tangles: A new connectivity measure and some applications

Bruce Reed
Tree Decomposition

\[ G=(V,E) \]

\[ T=(V_T, E_T) \]

\[ X_t = \{d, e, c\} \subseteq V \]

Example from Bodlaender’s talk
Tree Decomposition

\[ G = (V, E) \]

\[ T = (V_T, E_T) \]

- \( \bigcup_t X_t = V \)
- For each \( v \in V \), \( \{ t \mid v \in X_t \} \) form a (connected) sub-tree of \( T \)
- For each edge \( uv \in E \), exists \( t \) such that \( u, v \in X_t \)

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Treewidth

$G = (V, E)$

$T = (V_T, E_T)$

Width of decomposition := $\max_t |X_t|

\[ tw(G) = (\text{min width of a tree decomp for } G) - 1 \]
Examples

\[ \text{tw(Tree)} = 1 \]

\[ \text{tw(Cycle)} = 2, \quad \text{tw}(G) \leq 2 \Leftrightarrow G \text{ is series-parallel} \]

Figure 1: The recursive diamond graphs of order 0, 1, 2, and 3.

\[ V(H_3) = \emptyset. \] Since we chose \( Q \) to be minimum, \( u \) and \( v \) do not belong to the same \( H_i \), \( 1 \leq i \leq 4 \). If \( u \in V(H_2) \cup V(H_4) \), then the extremities of \( H_1 \) are a \( u-v \)-vertex cut of size 2 in \( G \mid Q \) and in \( G \). Otherwise, suppose, without loss of generality, that \( u \in V(H_1) \cap V(H_2) \). Since \( v \in V(H_1) \cup V(H_2) \), the other two extremities of \( H_1 \) and \( H_2 \) form a \( u-v \)-vertex cut \( C \) of size 2 in \( G \mid Q \). The set \( C \) is also a \( u-v \)-vertex cut in \( G \), unless \( q < p \) and \( u \) is an extremity of another subgraph \( J \) of \( G \) isomorphic to \( G_q \) that is edge-disjoint from \( G \mid Q \). In the latter case, add the other extremity of \( J \) to \( C \) to obtain a \( u-v \)-vertex cut in \( G \) of size 3.

Since \( G \) has a \( u-v \)-vertex cut of size at most 3, by Menger's theorem \([6]\), there are at most 3 independent \( u-v \) paths in \( G \).

An application Lemma 1 has been used in an algorithm \([2]\) for the detection of backdoor sets to ease Satisfiability solving. A backdoor set of a propositional formula is a set of variables such that assigning truth values to the variables in the backdoor set moves the formula into a polynomial-time decidable class; see \([3]\) for a survey. The class of nested formulas was introduced by Knuth \([5]\) and their satisfiability can be decided in polynomial time. To find a backdoor set to the class of nested formulas, the algorithm from \([2]\) considers the clause-variable incidence graph of the formula. If the formula is nested, this graph does not contain a \( K_{2,3} \)-minor with the additional property that the independent set of size 3 is obtained by contracting 3 connected subgraphs containing a variable each. In the correctness proof of the algorithm it is shown that in certain cases the formula does not have a small backdoor set.

This is shown by exhibiting two vertices \( u, v \) and 3 independent \( u-v \) paths in an auxiliary graph using Lemma 1. Expanding these edges to the paths they represent in the formula's incident graph gives rise to a \( K_{2,3} \)-minor with the desired property.

On the other hand, Lemma 2 shows the limitations of this approach if we would like to enlarge the target class to more general formulas.

Acknowledgment

We thank Chandra Chekuri for bringing the recursive diamond graphs to our attention \([1]\), and we thank Herbert Fleischner for valuable discussions on an earlier version of this note.

References


Figure from Serge Gasper's paper.
Examples

tw($K_n$) = n - 1
Examples

- $k \times k$ grid: $\text{tw}(G) = k - 1$

- $\text{tw}(G) = O(n^{1/2})$ for any planar $G$
Examples

- \( k \times k \) wall: \( tw(G) = \Theta(k) \)
Examples

Expander: \( \text{tw}(G) = \Theta(n) \)

Vertex expander: \( |\Pi(S)| \geq \alpha |S| \) for all \( |S| \leq n/2 \)

Edge expander: \( |\delta(S)| \geq \alpha |S| \) for all \( |S| \leq n/2 \)
and max degree \( \Delta = O(1) \)

Recall treewidth of complete graph = \( n-1 \)
Complexity of Treewidth

[Arnborg-Corneil-Proskurowski’87]

Given $G, k$ checking if $\text{tw}(G) \leq k$ is NP-Complete

[Bodleander’93]

For fixed $k$, linear time algorithm to check if $\text{tw}(G) \leq k$
Complexity of Treewidth

$\alpha$-approx. for node separators implies $O(\alpha)$-approx. for treewidth

[Feige-Hajiaghayi-Lee’05]

Polynomial time algorithm to output tree decomposition of width $O(tw(G) \log^{1/2} tw(G))$

[Arora-Rao-Vazirani’04] algorithm adapted to node separators
Connection to separators

$X_t \cap X_{t'} = \{a,f\}$ is a separator

$\text{tw}(G) \leq k$ implies $G$ can be recursively partitioned via “balanced” separators of size $k$
Connection to separators

\[ \text{tw}(G) \leq k \text{ implies } G \text{ can be recursively partitioned via "balanced" separators of size } k \]

**Approximate converse:** \( \text{tw}(G) > k \) implies some set of size \( \Omega(k) \) which has no balanced separator of size \( \leq k \)
A set $X \subseteq V$ is well-linked in $G$ if for all $A, B \subseteq X$ there are $\min(|A|, |B|)$ node-disjoint $A$-$B$ paths.
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A set $X \subseteq V$ is well-linked in $G$ if for all $A, B \subseteq X$ there are $\min(|A|,|B|)$ node-disjoint $A$-$B$ paths.

No sparse node-separators for $X$ in $G$. 

Diagram shows a set $X$ partitioned into subsets $A$, $B$, and $C$. The diagram illustrates the concept of well-linked sets with node-disjoint paths between subsets. 

The text is clearly and concisely presented, making it easy to understand the concept of well-linked sets in graph theory.
Treewidth & Well-linked Sets

A set $X \subseteq V$ is well-linked in $G$ if for all $A, B \subseteq X$ there are $\min(|A|, |B|)$ node-disjoint $A$-$B$ paths

No sparse node-separators for $X$ in $G$

$wl(G) = \text{cardinality of the largest well-linked set in } G$

**Known Theorem:** $wl(G) \leq tw(G) \leq 4 \cdot wl(G)$
A set $X \subseteq V$ is **well-linked** in $G$ if for all $A, B \subseteq X$ there are $\min(|A|, |B|)$ node-disjoint $A$-$B$ paths.

No sparse node-separators for $X$ in $G$.

$G$ is a “vertex expander” for the subset $X$. 

**Treewidth & Well-linked Sets**
Treewidth “paradigm”

- If G has “small” (constant) treewidth, solve problem efficiently or approximately. Running time typically depends exponentially on \( tw(G) \)

- If G has “large” treewidth use structure, in particular, obstructions such as grids
Structure of graphs with “large” treewidth

How large can a graph’s treewidth be?
  • for specific classes of graphs, say planar graphs?

What can we say about a graph with “large” treewidth?

“Large”: \( tw(G) = n^\delta \) where \( \delta \in (0,1) \)
Graph Minors

$H$ is a minor of $G$ if it is obtained from $G$ by

- edge and vertex deletions
- edge contractions

$G$ is minor closed family of graphs if for each $G \in G$ all minors of $G$ are also in $G$

- planar graphs,
- genus $\leq g$ graphs,
- $tw \leq k$ graphs
Graph Minors

[Kuratowski, Wagner]:

\( G \) is planar iff \( G \) excludes \( K_5 \) and \( K_{3,3} \) as a sub-division/minor.

\( H \) a fixed graph

\( \mathcal{G}_H = \{ G \mid G \text{ excludes } H \text{ as a minor} \} \)

\( \mathcal{G}_H \) is minor closed

What is the structure of \( \mathcal{G}_H \)?
Theorem: $\text{tw}(G) \geq f(k)$ implies $G$ contains a clique minor of size $k$ or a grid minor of size $k$. 

Robertson-Seymour Grid-Minor Theorem(s)
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Theorem: $\text{tw}(G) \geq f(k)$ implies $G$ contains a clique minor of size $k$ or a grid minor of size $k$.

Corollary (Grid-minor theorem): $\text{tw}(G) \geq f(k)$ implies $G$ contains a grid minor of size $k$.

Previous best bound: $f(k) = 2^{O(k^2 \log k)}$ [Kawarabayashi-Kobayashi'12, Leaf-Seymour'12]

$\text{tw}(G) \geq h$ implies grid minor of size at least $(\log h)^{1/2}$.
Robertson-Seymour Grid-Minor Theorem(s)

**Theorem:** $\text{tw}(G) \geq f(k)$ implies $G$ contains a clique minor of size $k$ or a grid minor of size $k$

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Previous best bound: $f(k) = 2^{O(k^2 \log k)}$

$\text{tw}(G) \geq h$ implies grid minor of size at least $(\log h)^{1/2}$

**Theorem (Robertson-Seymour-Thomas):** $G$ is planar implies grid minor of size $\Omega(\text{tw}(G))$
Conjecture on Grid-Minors

tw(G) \geq k \text{ implies } G \text{ has grid-minor of size } \text{poly}(k)

Examples show grid-minor cannot be larger than \( O(\sqrt{(k/\log k)}) \) [Robertson-Seymour-Thomas]

**Theorem: [Reed-Wood’??]** tw(G) \geq k \text{ implies } G \text{ has grid-like minor of size } \Omega(k^{1/4}/\log k)
Recent Results

[C-Chuzhoy’13]

**Theorem:** $\text{tw}(G) \geq k$ implies that $G$ has a grid-minor of size $\Omega(k^{\delta})$ where $\delta \geq 1/98 - o(1)$

**Corollary:** If $G$ excludes a planar graph $H$ as a minor then $\text{tw}(G) \leq |V(H)|^c$ for some fixed constant $c$
Other Recent Results

- Large routing structures in large treewidth graphs
  - applications to approximating disjoint paths problems
- Treewidth decomposition theorems
  - applications to fixed parameter tractability
  - applications to Erdos-Posa type theorems
- Some related results and applications
Treewidth and Routing

[Chuzhoy’11, Chuzhoy-Li’12]

If $tw(G) \geq k$ then there is an expander of size $k/polylog(k)$ that can be “embedded” into $G$ with edge congestion 2

Conjectured by [C-Khanna-Shepherd’05]

Together with previous tools/ideas, $polylog(k)$ approximation with $O(1)$ congestion for maximum edge disjoint paths problems
If $tw(G) \geq k$ then there is an expander of size $k/polylog(k)$ that can be "embedded" into $G$ with $O(1)$ node congestion.

Conjectured by [C-Khanna-Shepherd’05]

Together with previous tools/ideas, $polylog(k)$ approximation with $O(1)$ congestion for maximum node disjoint paths problems.

[C-Chuzhoy’14] Congestion can be improved to $2$?
A Key Tool

[Chuzhoy’11]
A graph decomposition algorithm

[C-Chuzhoy’12]
Abstract and improve to obtain treewidth decomposition results and applications
Treewidth Decomposition

Let $tw(G) = k$. Given integers $h$ and $r$ want to partition $G$ into node disjoint graphs $G_1, G_2, ..., G_r$ such that $tw(G_i) \geq h$ for all $i$. 

Diagram:

- $G$ with $tw(G) = k$
- $G_1$, $G_2$, ..., $G_r$ with $tw(G_i) \geq h$
Theorem(s): Let $tw(G) = k$. Then $G$ can be partitioned into node disjoint graphs $G_1, G_2, \ldots, G_h$ such that $tw(G_i) \geq r$ for all $i$ if

- $h r^2 \leq k / \text{polylog}(k)$ or
- $h^3 r \leq k / \text{polylog}(k)$
Theorem(s): Let $\text{tw}(G) = k$. Then $G$ can be partitioned into node disjoint graphs $G_1, G_2, \ldots, G_r$ such that $\text{tw}(G_i) \geq h$ for all $i$ if

- $r h^2 \leq k / \text{polylog}(k)$ or
- $r^3 h \leq k / \text{polylog}(k)$

Conjecture: sufficient if $h r \leq k / \text{polylog}(k)$

Examples show that $h r \leq k / \log(k)$ is necessary
Theorem(s): Let $\text{tw}(G) = k$. Then $G$ can be partitioned into node disjoint graphs $G_1, G_2, ..., G_r$ such that $\text{tw}(G_i) \geq h$ for all $i$ if

- $r h^2 \leq k/\text{polylog}(k)$ or
- $r^3 h \leq k/\text{polylog}(k)$

Remark: Grid-minor theorem implies a treewidth decomposition theorem but with weaker bounds. New grid-minor theorem built upon the decomposition result
Some Details of Grid-Minor Construction

**Theorem:** $tw(G) \geq k$ implies that $G$ has a grid-minor of size $\Omega(k^\delta)$ where $\delta \geq 1/98$
Any graph of treewidth $k$

Nice graph of treewidth $k/	ext{polylog}(k)$

A family of $r$ disjoint good routers

Weak Tree-of-sets-system

Tree-of-sets-system

Path-of-sets system

Large grid minor
Path-of-Sets System
Each \( C_i \) is a connected cluster.
The clusters are disjoint.
Every consecutive pair of clusters connected by \( h \) paths.
All blue paths are disjoint from each other and internally disjoint from the clusters.
The interface vertices are well-linked inside $C_i$. 

Interface vertex
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The interface vertices are well-linked inside $C_i$. 
**Theorem:** Given a path-of-sets system, there is an efficient algorithm to find a grid minor of size $\Omega(\sqrt{h})$.

**Corollary:** enough to find a path-of-sets system with $h = \text{poly}(k)$, where $k$ is the treewidth.
Relation to \((h,r)\)-grill

**Thm [Leaf, Seymour ‘12]**: Given a \((h,r)\)-grill efficient algorithm to find a grid-minor.

\((h,r)\) grills very closely related to \((h,r)\)-path-of-sets-systems

Our proof is essentially identical to theirs but path-of-sets-system gives slightly better parameters.
From Path-of-Sets System to Grid Minor
Building the Grid
Building the Grid
Building the Grid
Building the Grid
For each $C_i$, we’ll be looking for a direct path connecting some consecutive pair of horizontal paths.
Tree of Sets System

Degree 3 tree $T$ on $r$ nodes
$h$ disjoint paths for each edge of $T$

Interface is well-linked
Tree to Path

Easy if tree has a long path

Main case: tree has many leaves
High-Level Idea

Stage 1: connect every leaf to the root by many disjoint paths
Stage 2: exploit these paths to build a path-of-sets system
Stage 2
Stage 2
Stage 2
Stage 2
Stage 2
Stage 2
On Well-linked Sets
A set $X \subseteq V$ is **well-linked** in $G$ if for all $A, B \subseteq X$ there are $\min(|A|, |B|)$ node-disjoint $A$-$B$ paths.

$X \subseteq V$ is **edge-well-linked** if paths are edge-disjoint.
Fractionally Well-linked Sets

For $\alpha \in (0,1]$ a set $X \subseteq V$ is $\alpha$-well-linked in $G$ if for all $A, B \subseteq X$ there are $\min(|A|,|B|)$ $A$-$B$ paths with edge/node congestion $1/\alpha$. 
Grouping/Boosting Results on Well-linked Sets

[C-Khanna-Shepherd’05]

Given promise that $X \subseteq V$ is $\alpha$-edge-well-linked in $G$ can efficiently find $X' \subseteq X$ such that $X'$ is edge-well-linked in $G$ and

$$|X'| = \Omega(\alpha |X|)$$

Similar guarantee for node-well-linkedness but $X'$ was only $\frac{1}{4}$-node-well-linked
Grouping/Boosting Results on Well-linked Sets

[C-Chuzhoy’13]

• $X \subseteq V$ is $\alpha$-edge-well-linked in $G$
• $X$ is partitioned into $X_1, X_2, ..., X_r$
• $\Delta(G)$ is maximum degree of $G$

Exists $X' \subseteq X$ such that

• $X'$ is node-well-linked in $G$
• for $1 \leq j \leq r$, $|X' \cap X_j| = \Omega(\alpha|X_j|/\Delta^2(G))$

Efficiency results in extra log factor loss.
Decomposition and Tree of Sets

Mostly from [Chuzhoy’11, Chuzhoy-Li’12] + node-connectivity ideas from [C-Ene’13]
Treewidth Decomposition

- \( r h^2 \leq k / \text{polylog}(k) \) or
- \( r^3 h \leq k / \text{polylog}(k) \)
A Family of Good Routers

- \( r \) disjoint clusters \( S_1, \ldots, S_r \)
- For each \( S_i \), the interface vertices are \( 1/r^2 \)-well-linked inside \( S_i \)

\[ r = k^\varepsilon \]
A Family of Good Routers

• $r$ disjoint clusters $S_1, \ldots, S_r$
• For each $S_i$, the interface vertices are $1/r^2$-well-linked inside $S_i$
A Family of Good Routers

- $r$ disjoint clusters $S_1, \ldots, S_r$
- For each $S_i$, the interface vertices are $1/r^2$-well-linked inside $S_i$
- For each $i, j$, there are $p = r^{20}$ node-disjoint paths connecting $S_i$ to $S_j$. They are linked.
From Good Family to Tree of Set System
Want: get rid of black vertices in $H$
Want: get rid of black vertices in $H$
Graph $H'$

- edge in $H' \Rightarrow$ direct path in $G$;
- degree of every vertex is $p/r$
- every pair of vertices has $p/r$ node-disjoint paths connecting them in $H'$

Use edge/element connectivity graph reduction steps [Mader, Hind-Oellerman, C-Korula]

$p = \Omega(r^{20})$
Graph $H'$

- edge in $H'$ $\rightarrow$ direct path in $G$;
- degree of every vertex is $p/r$;
- every pair of vertices has $p/r$ edge-disjoint paths connecting them in $H'$

If $v_i, v_j$ are connected by fewer than $p/r^4$ edges, delete all edges connecting them.
Graph $H'$

- edge in $H'$ $\rightarrow$ direct path in $G$;
- degree of every vertex is $p/r$;
- every pair of vertices has $p/r$ edge-disjoint paths connecting them in $H'$.

If $H'$ is a 1-tough graph, it has spanning tree of degree at most 3!

Need more work to obtain 1-tough graph by making degree uniform.

If $v_i, v_j$ are connected by fewer than $p/r^2$ edges, delete all edges connecting them.
Tree-of-Sets System
Tree-of-Sets System

\[ p = \Omega(r^{20}) \]

at least \( \frac{p}{r^4} \) edges
Tree-of-Sets System

\[ p = \Omega(r^{20}) \]

at least \( p/r^4 \) edges
Any graph of treewidth $k$

\[\text{Nice graph of treewidth } k/\text{polylog}(k)\]

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Open Problems

• Improve Grid-Minor theorem parameters
• Tight tradeoff for treewidth decomposition theorem
• Applications of related ideas?
• Similar results for directed treewidth?
Thank You!