Structure of Large-Treewidth Graphs: Recent Developments

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Joint work with
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Tree decompositions and Treewidth

Studied by [Halin'76]

Again by [Robertson & Seymour'84] as part of their graph minor project

Tree decompositions and Treewidth

Fundamental graph parameter

- key to graph minor theory
- generalizations to matroids via branchwidth
- algorithmic applications
- connections to theoretical computer science

Recent Progress on Approximation Algorithms for Disjoint Paths

[Chuzhoy'11] "Routing in Undirected Graphs with Constant Congestion"

[Chuzhoy-Li'12] "A Polylogarithmic Approximation for Edge-Disjoint Paths with Congestion 2"

[C-Ene'13] "Polylogarithmic Approximation for Maximum Node Disjoint Paths with Constant Congestion"

Answering a graph theoretic conjecture of [C-Khanna-Shepherd]

Some recent developments in treewidth

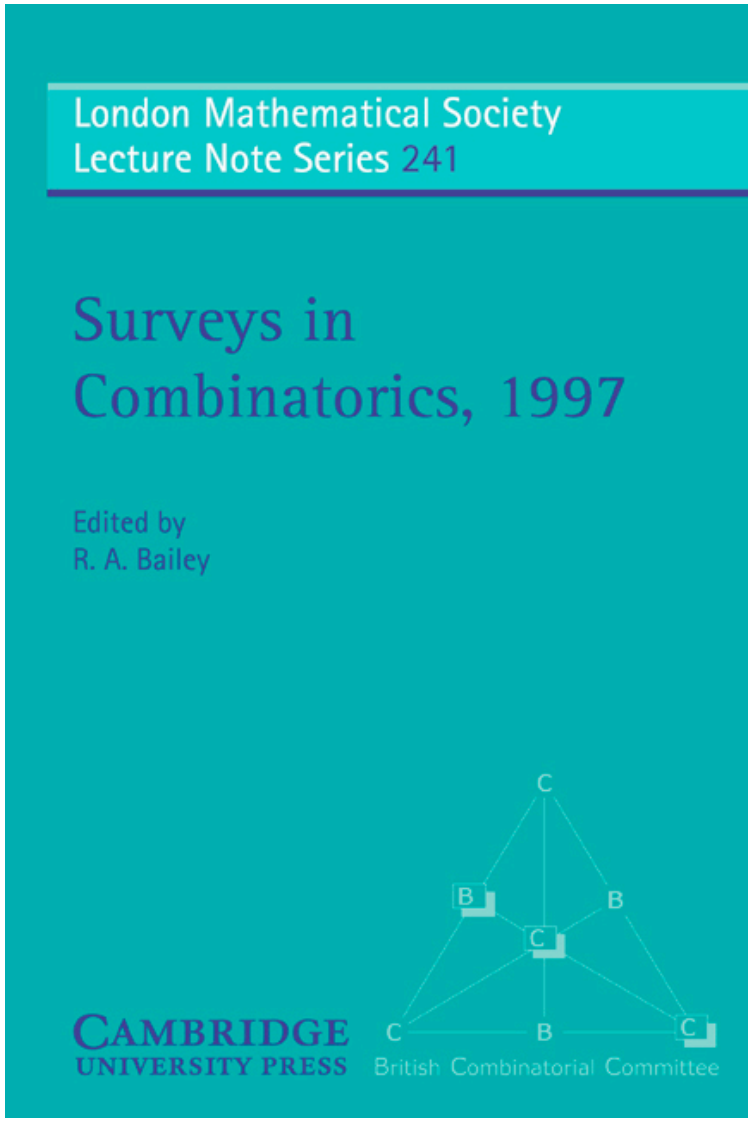
[C-Chuzhoy'12] "Large-treewidth graph decompositions and applications"

[C-Chuzhoy'13] "Polynomial bounds for the Grid-Minor Theorem"

Ongoing work

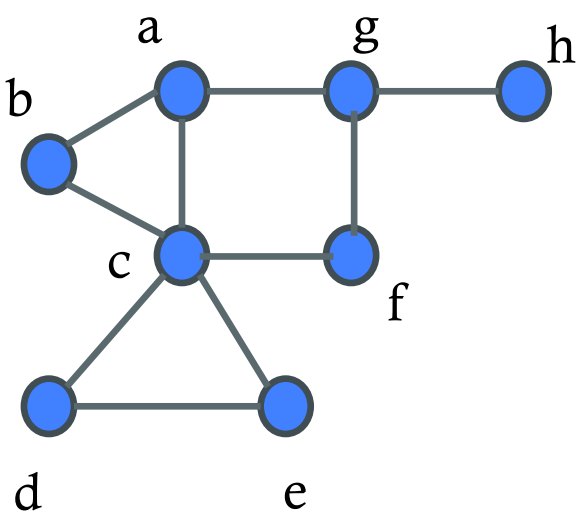
Tree width and tangles: A new connectivity measure and some applications

Bruce Reed

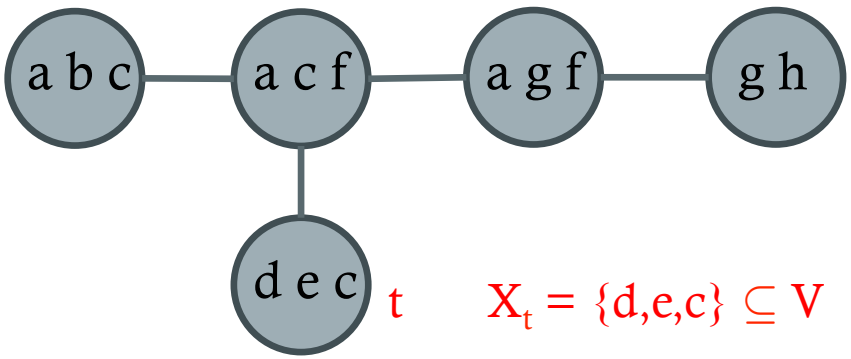


Tree Decomposition

$G=(V,E)$



$T=(V_T, E_T)$



$X_t = \{d,e,c\} \subseteq V$

Example from Bodlaender's talk

Tree Decomposition

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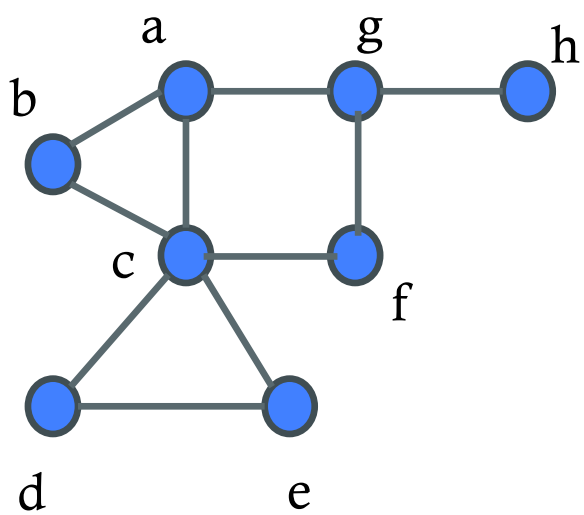

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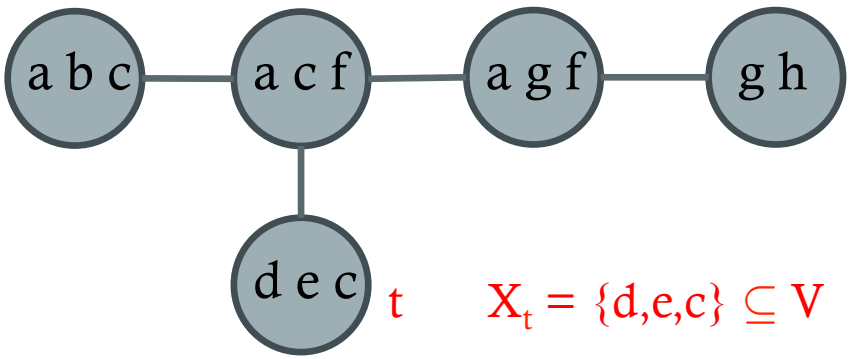

$X_t = \{d,e,c\} \subseteq V$

- $\cup_t X_t = V$
- For each $v \in V$, $\{ t \mid v \in X_t \}$ form a (connected) sub-tree of T
- For each edge $uv \in E$, exists t such that $u,v \in X_t$

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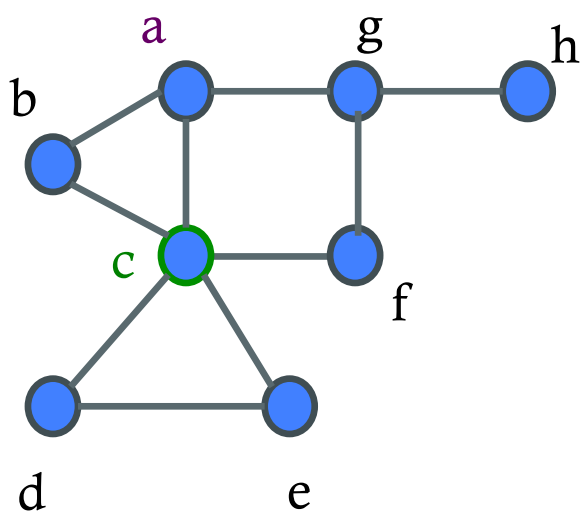

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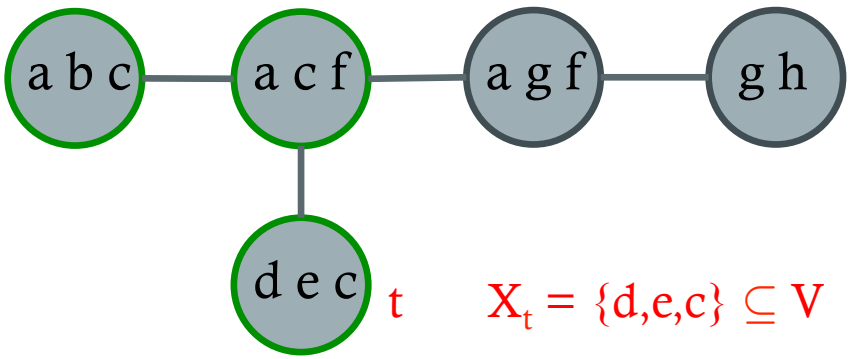

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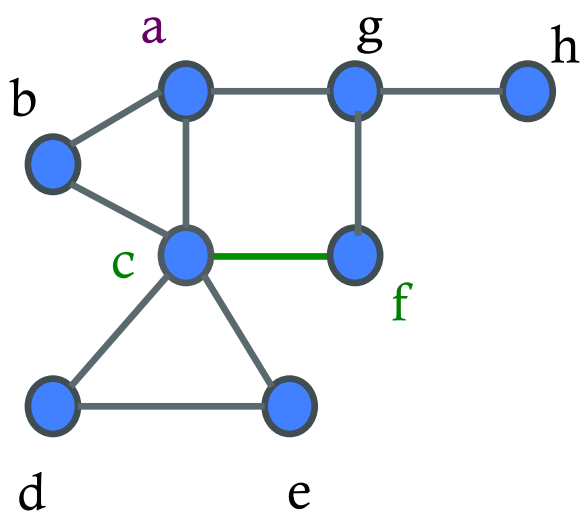

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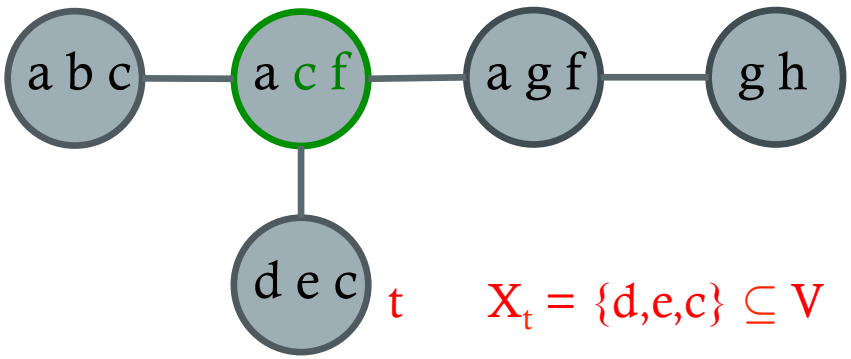

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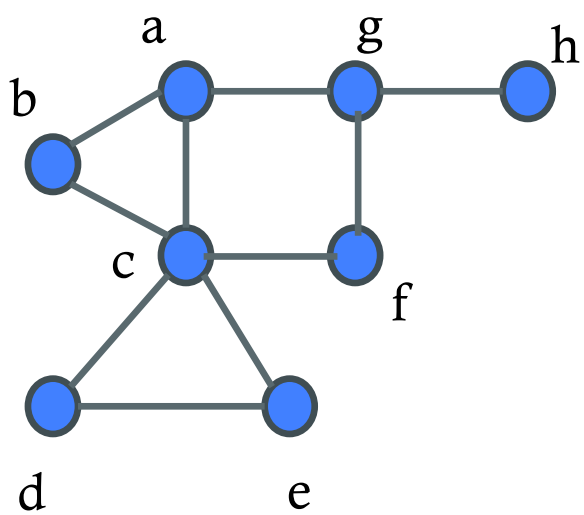
Treewidth

$G=(V,E)$



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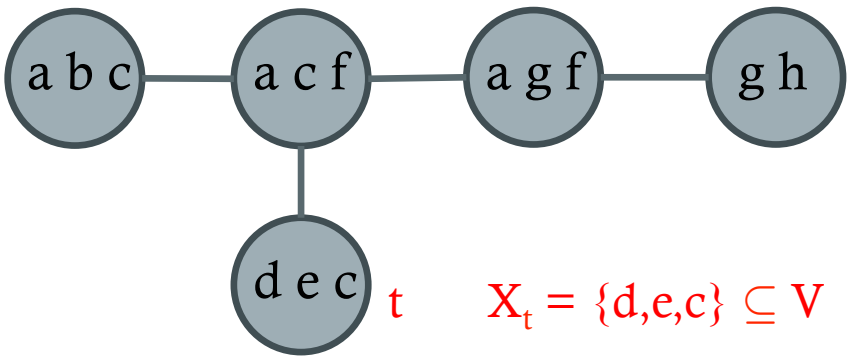


$X_t = \{d,e,c\} \subseteq V$

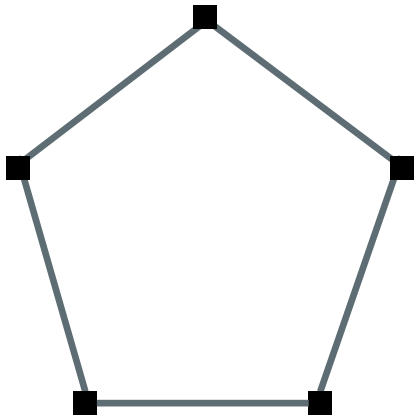
Width of decomposition := $\max_t |X_t|$

$tw(G) =$ (min width of a tree decomp for G) – 1

Examples

tw(Tree) = 1

tw(Cycle) = 2, tw(G) $\leq$ 2 $\Leftrightarrow$ G is series-parallel



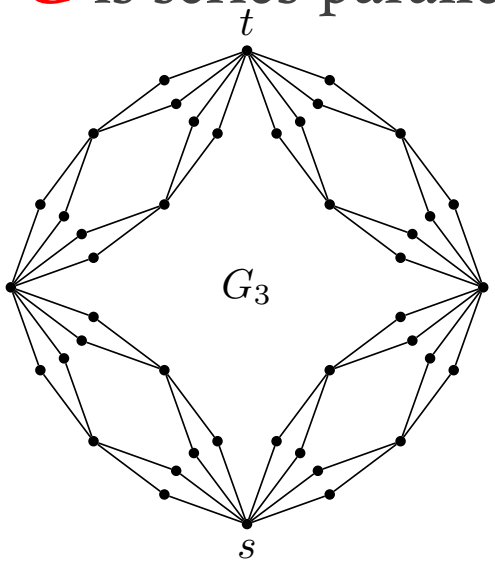
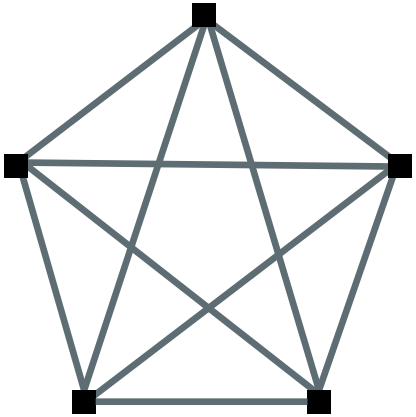


Figure from Serge Gasper's paper

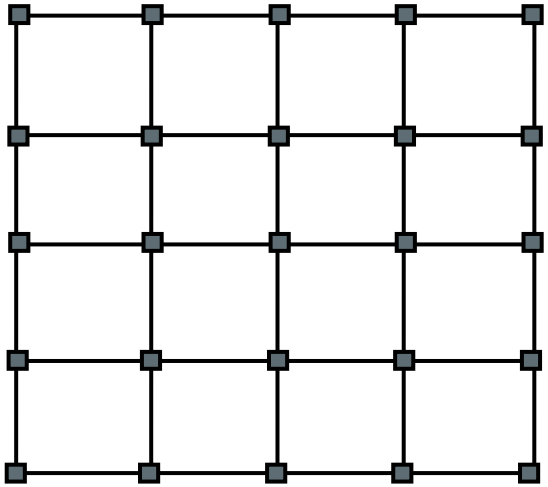
Examples

$tw(K_n) = n-1$



Examples

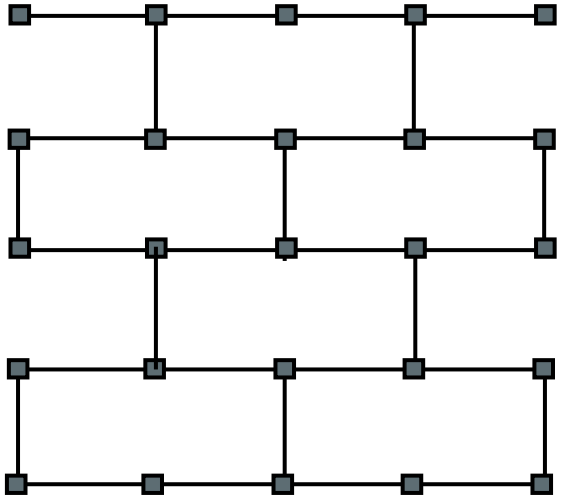
- $k \times k$ grid: $tw(G) = k-1$



- $tw(G) = O(n^{1/2})$ for any planar G

Examples

- k x k wall: $tw(G) = \Theta(k)$



Examples

Expander: $tw(G) = \Theta(n)$

Vertex expander: $|\Gamma(S)| \geq \alpha\ |S|$ for all $|S| \leq n/2$

Edge expander: $|\delta(S)| \geq \alpha\ |S|$ for all $|S| \leq n/2$
and max degree $\Delta = O(1)$

Recall treewidth of complete graph = $n-1$

Complexity of Treewidth

[Arnborg-Corneil-Proskurowski'87]

Given G, k checking if $tw(G) \leq k$ is NP-Complete

[Bodleander'93]

For fixed k, linear time algorithm to check if $tw(G) \leq k$

Complexity of Treewidth

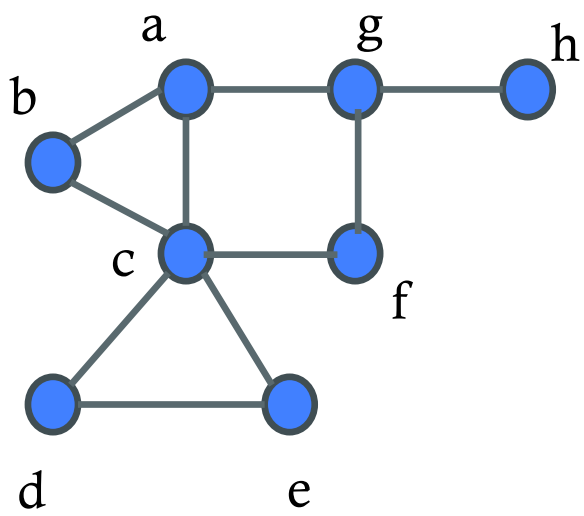
α-approx. for node separators implies $O(\alpha)$-approx. for treewidth

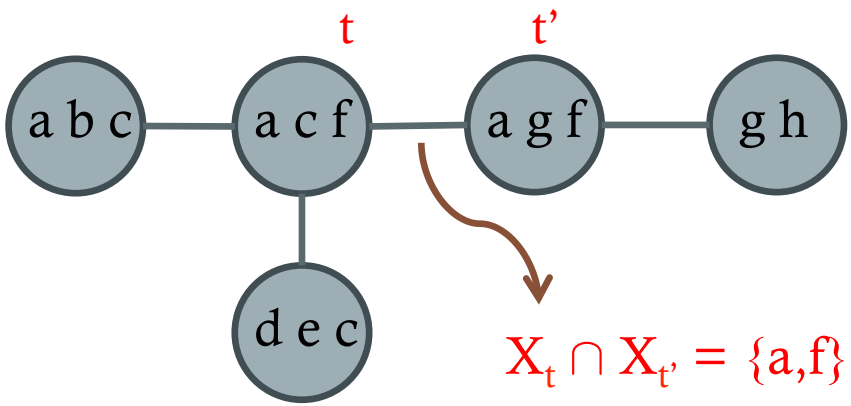
[Feige-Hajiaghayi-Lee'05]

Polynomial time algorithm to output tree decomposition of width $O(tw(G) \log^{1/2} tw(G))$

[Arora-Rao-Vazirani'04] algorithm adapted to node separators

Connection to separators

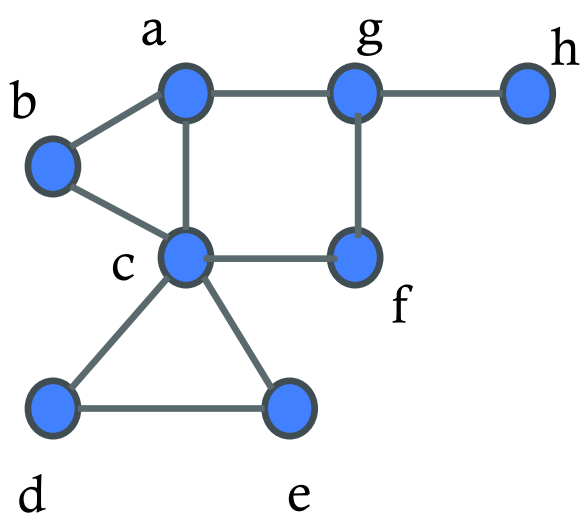


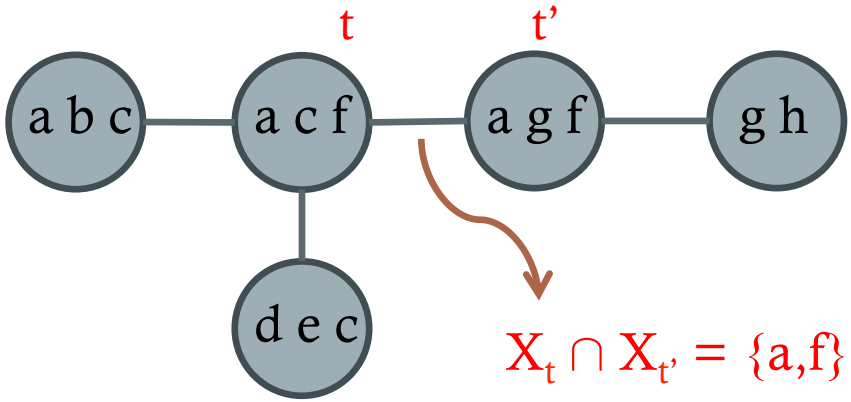


$X_t \cap X_{t'} = \{a,f\}$ is a separator

$tw(G) \leq k$ implies G can be recursively partitioned via "balanced" separators of size k

Connection to separators





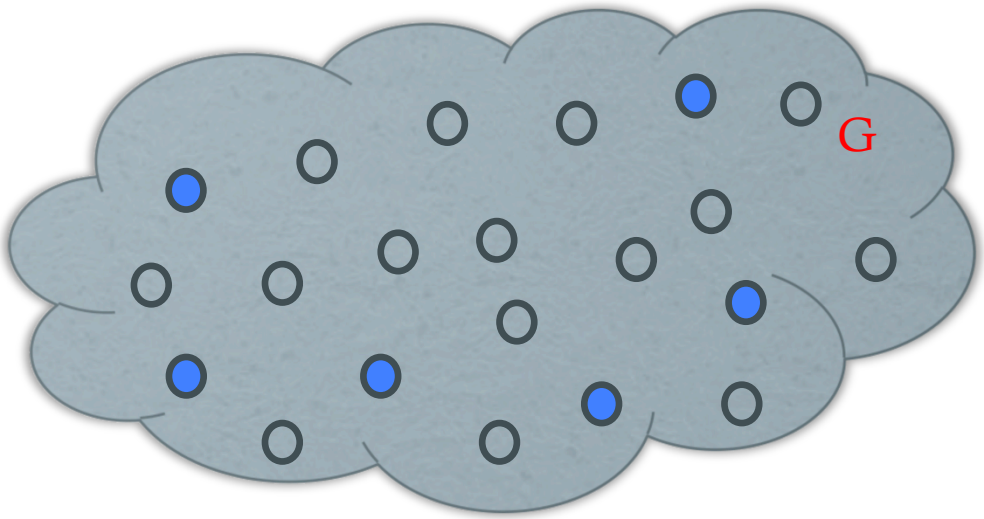
$X_t \cap X_{t'} = \{a,f\}$ is a separator

$tw(G) \leq k$ implies G can be recursively partitioned via "balanced" separators of size k

Approximate converse: $tw(G) > k$ implies some set of size $\Omega(k)$ which has no balanced separator of size $\leq k$

Well-linked Sets

A set $X \subseteq V$ is **well-linked** in G if for all $A, B \subseteq X$ there are $\min(|A|,|B|)$ node-disjoint A-B paths



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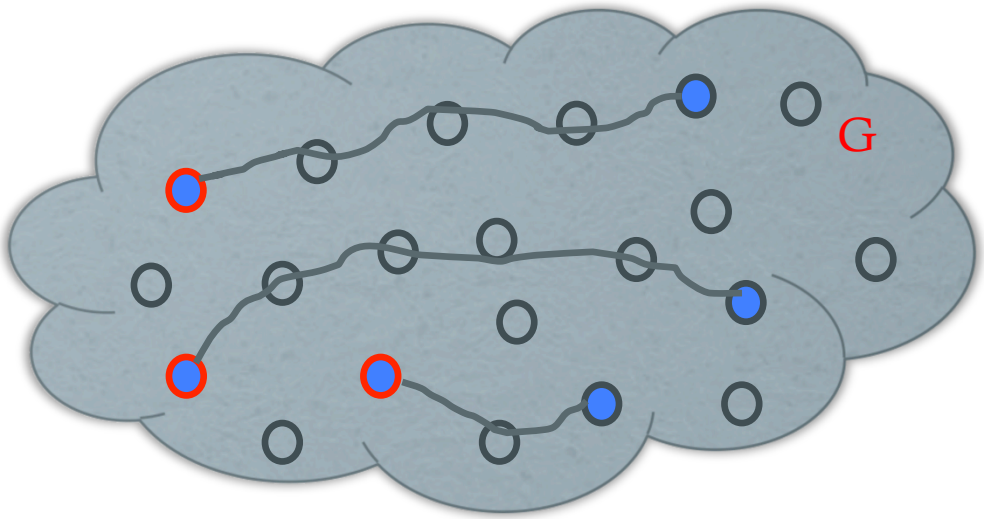

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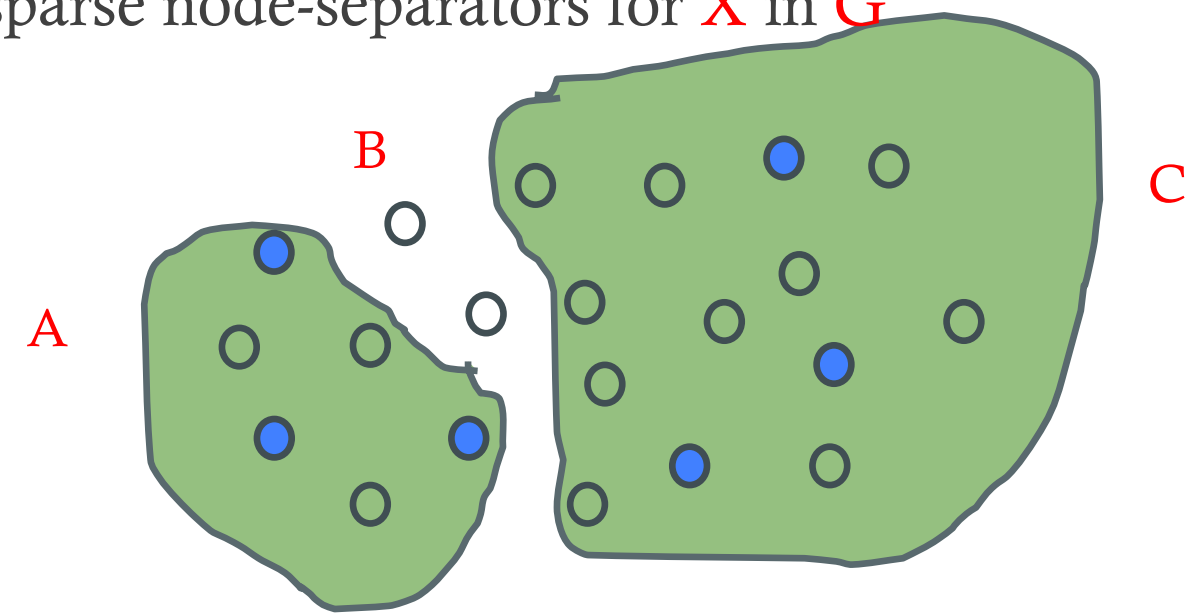
No sparse node-separators for X in G



Treewidth & Well-linked Sets

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$wl(G)$ = cardinality of the largest well-linked set in G

Known Theorem: $wl(G) \leq tw(G) \leq 4\, wl(G)$

Treewidth & Well-linked Sets

A set $X \subseteq V$ is **well-linked** in G if for all $A, B \subseteq X$ there are $\min(|A|,|B|)$ node-disjoint A-B paths

No sparse node-separators for X in G

G is a "vertex expander" for the subset X

Treewidth “paradigm”

- If G has “small” (constant) treewidth, solve problem efficiently or approximately. Running time typically depends exponentially on tw(G)
- If G has “large” treewidth use structure, in particular, obstructions such as grids

Structure of graphs with "large" treewidth

How large can a graph's treewidth be?

- for specific classes of graphs, say planar graphs?

What can we say about a graph with "large" treewidth?

"Large": $tw(G) = n^{\delta}$ where $\delta \in (0,1)$

Graph Minors

H is a minor of G if it is obtained from G by

- edge and vertex deletions
- edge contractions

$\mathcal{G}$ is minor closed family of graphs if for each $G \in \mathcal{G}$ all minors of G are also in $\mathcal{G}$

- planar graphs,
- genus $\leq$ g graphs,
- tw $\leq$ k graphs

Graph Minors

[Kuratowski, Wagner]:

G is planar iff G excludes K_5 and $K_{3,3}$ as a sub-division/ minor.

H a fixed graph

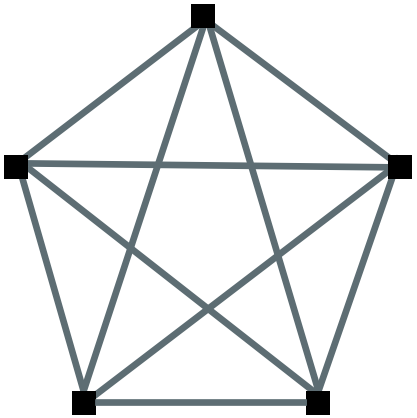
$\mathcal{G}_H = \{G \mid G \text{ excludes } H \text{ as a minor}\}$

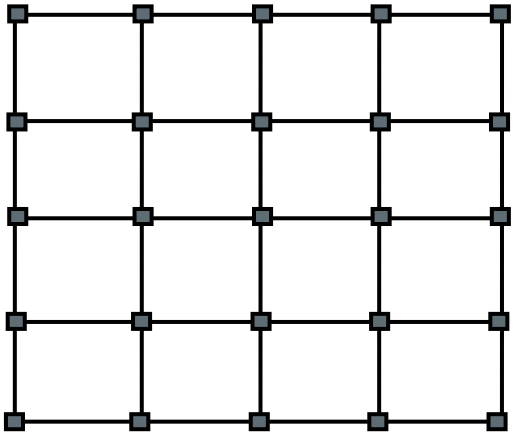
$\mathcal{G}_H$ is minor closed

What is the structure of $\mathcal{G}_H$?

Robertson-Seymour Grid-Minor Theorem(s)

Theorem: $tw(G) \geq f(k)$ implies G contains a clique minor of size k or a grid minor of size k





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Corollary (Grid-minor theorem): $tw(G) \geq f(k)$ implies G contains a grid minor of size k

Previous best bound: $f(k) = 2^{O(k^2 \log k)}$ [Kawarabayashi-Kobayashi'12, Leaf-Seymour'12]

$tw(G) \geq h$ implies grid minor of size at least $(\log h)^{1/2}$

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Theorem (Robertson-Seymour-Thomas): G is *planar* implies grid minor of size $\Omega(tw(G))$

Conjecture on Grid-Minors

$tw(G) \geq k$ implies G has grid-minor of size $poly(k)$

Examples show grid-minor cannot be larger than $O(\sqrt{(k/\log k)})$ [Robertson-Seymour-Thomas]

Theorem: [Reed-Wood'??] $tw(G) \geq k$ implies G has *grid-like* minor of size $\Omega(k^{1/4}/\log k)$

Recent Results

[C-Chuzhoy'13]

Theorem: $tw(G) \geq k$ implies that G has a grid-minor of size $\Omega(k^{\delta})$ where $\delta \geq 1/98 - o(1)$

Corollary: If G excludes a planar graph H as a minor then $tw(G) \leq |V(H)|^{c}$ for some fixed constant c

Other Recent Results

- Large routing structures in large treewidth graphs
  - applications to approximating disjoint paths problems
- Treewidth decomposition theorems
  - applications to fixed parameter tractability
  - applications to Erdos-Posa type theorems
- Some related results and applications

Treewidth and Routing

[Chuzhoy'11,Chuzhoy-Li'12]

If $tw(G) \geq k$ then there is an expander of size $k/polylog(k)$ that can be "embedded" into G with edge congestion 2

Conjectured by [C-Khanna-Shepherd'05]

Together with previous tools/ideas, polylog(k) approximation with O(1) congestion for maximum edge disjoint paths problems

Treewidth and Routing

[Chuzhoy'11,Chuzhoy-Li'12] plus [C-Ene'13]

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Together with previous tools/ideas, polylog(k) approximation with O(1) congestion for maximum node disjoint paths problems

[C-Chuzhoy'14] Congestion can be improved to 2?

A Key Tool

[Chuzhoy'11]

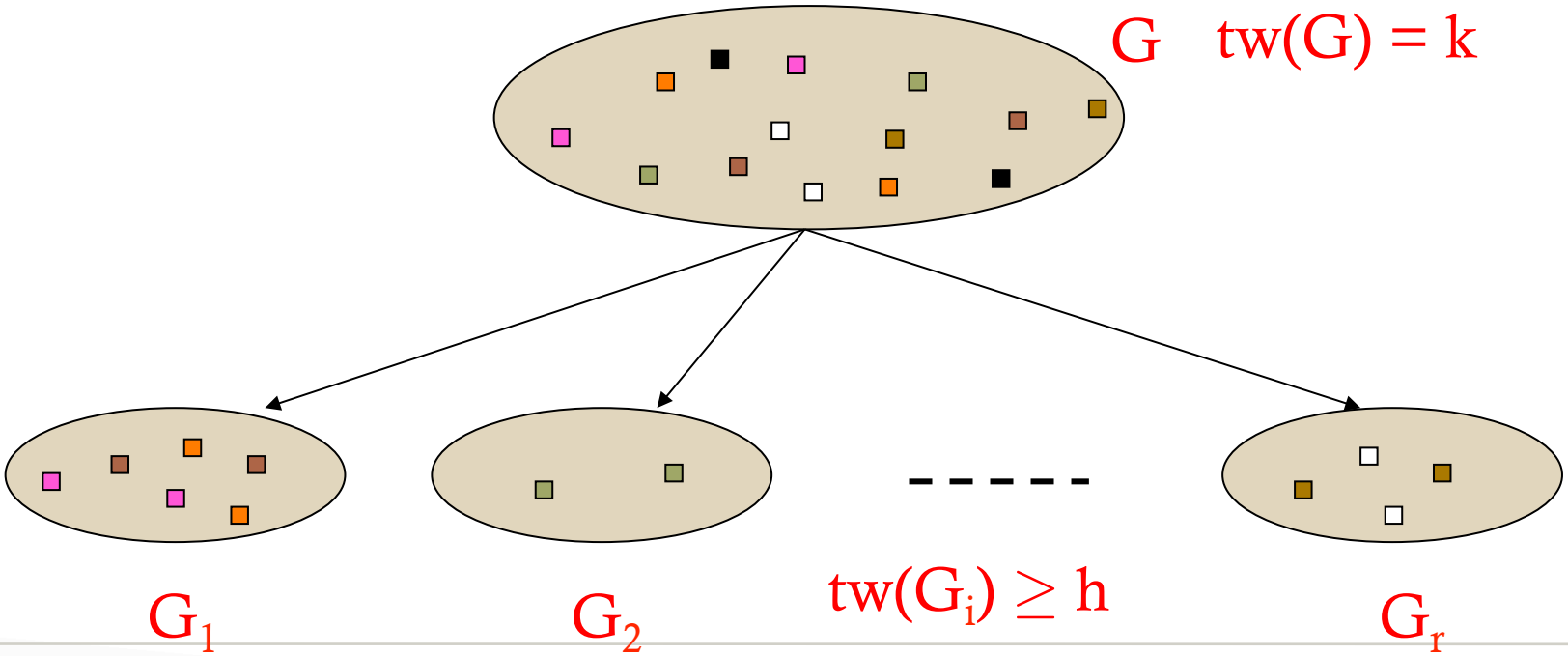
A graph decomposition algorithm

[C-Chuzhoy'12]

Abstract and improve to obtain treewidth decomposition results and applications

Treewidth Decomposition

Let $tw(G) = k$. Given integers h and r want to partition G into node disjoint graphs $G_1, G_2, \ldots, G_r$ such that $tw(G_i) \geq h$ for all i



Treewidth Decomposition Theorems

[C-Chuzhoy'12]

Theorem(s): Let $tw(G) = k$. Then G can be partitioned into node disjoint graphs $G_1, G_2, \ldots, G_h$ such that $tw(G_i) \geq r$ for all i if

- $h\, r^2 \leq k/\text{polylog}(k)$ or
- $h^3\, r \leq k/\text{polylog}(k)$

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Conjecture: sufficient if $h\, r \leq k/\text{polylog}(k)$

Examples show that $h\, r \leq k/\log(k)$ is necessary

Treewidth Decomposition Theorems

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Remark: Grid-minor theorem implies a treewidth decomposition theorem but with weaker bounds. New grid-minor theorem built upon the decomposition result

Some Details of Grid-Minor Construction

Theorem: $tw(G) \geq k$ implies that G has a grid-minor of size $\Omega(k^{\delta})$ where $\delta \geq 1/98$

Any graph of treewidth k

↓

Nice graph of treewidth k/polylog(k)

↓

A family of r disjoint good routers

↓

Weak Tree-of-sets-system

↓

Tree-of-sets-system

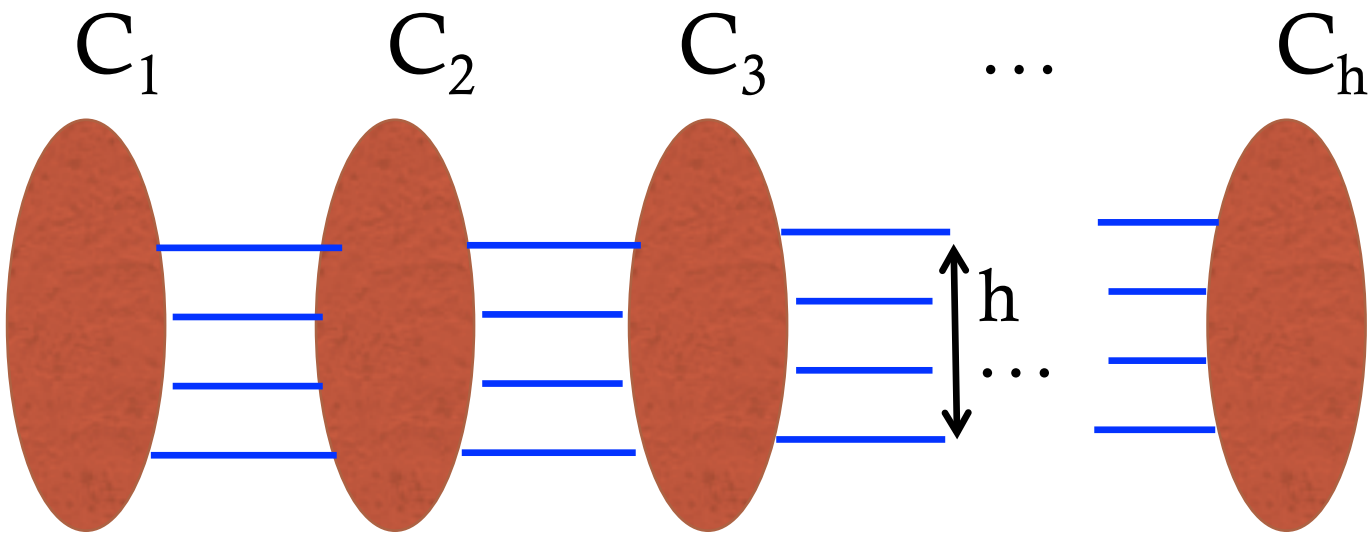
↓

Path-of-sets system

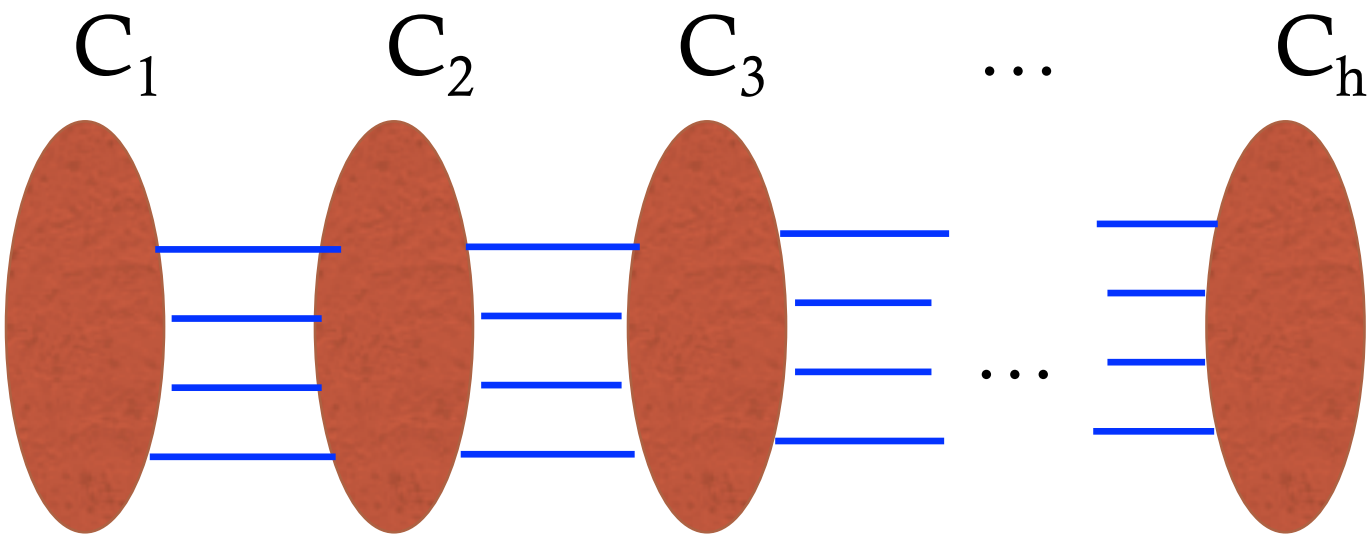
↓

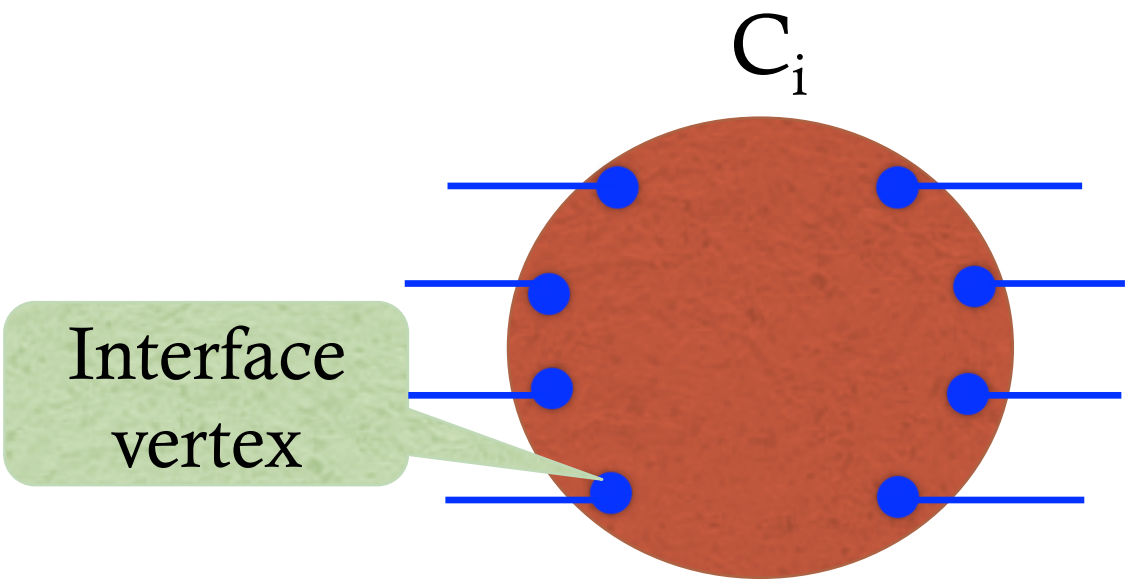
Large grid minor

Path-of-Sets System

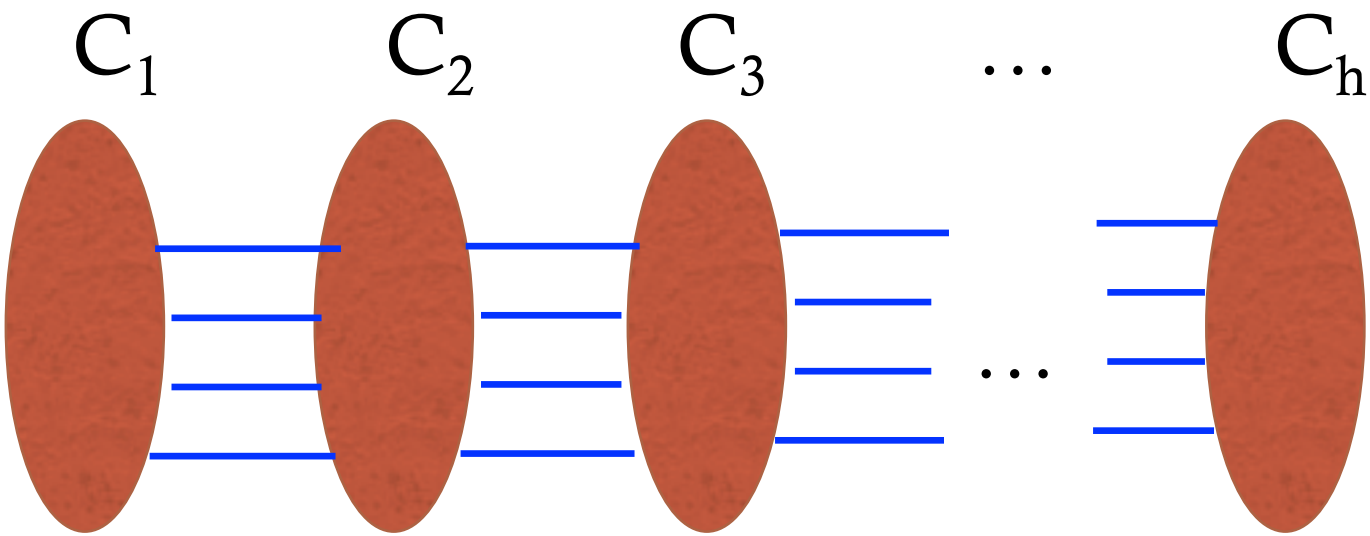


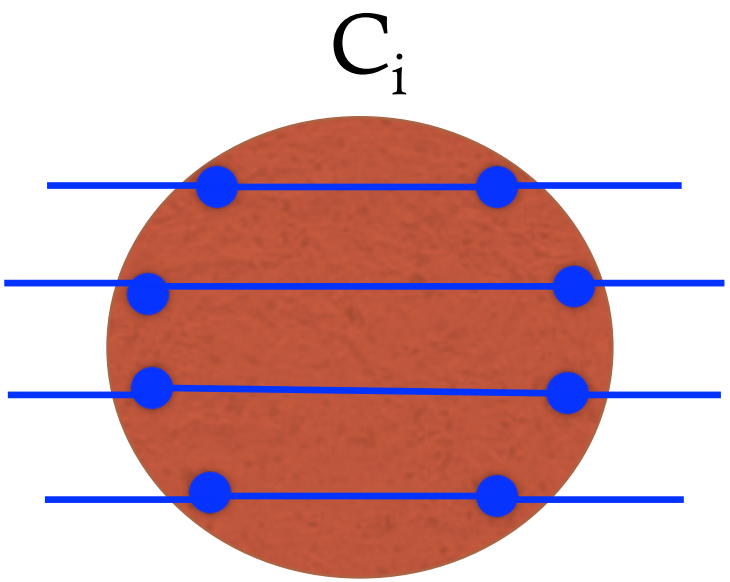
- Each C_i is a connected cluster
- The clusters are disjoint
- Every consecutive pair of clusters connected by h paths
- All blue paths are disjoint from each other and internally disjoint from the clusters





The interface vertices are well-linked inside C_i





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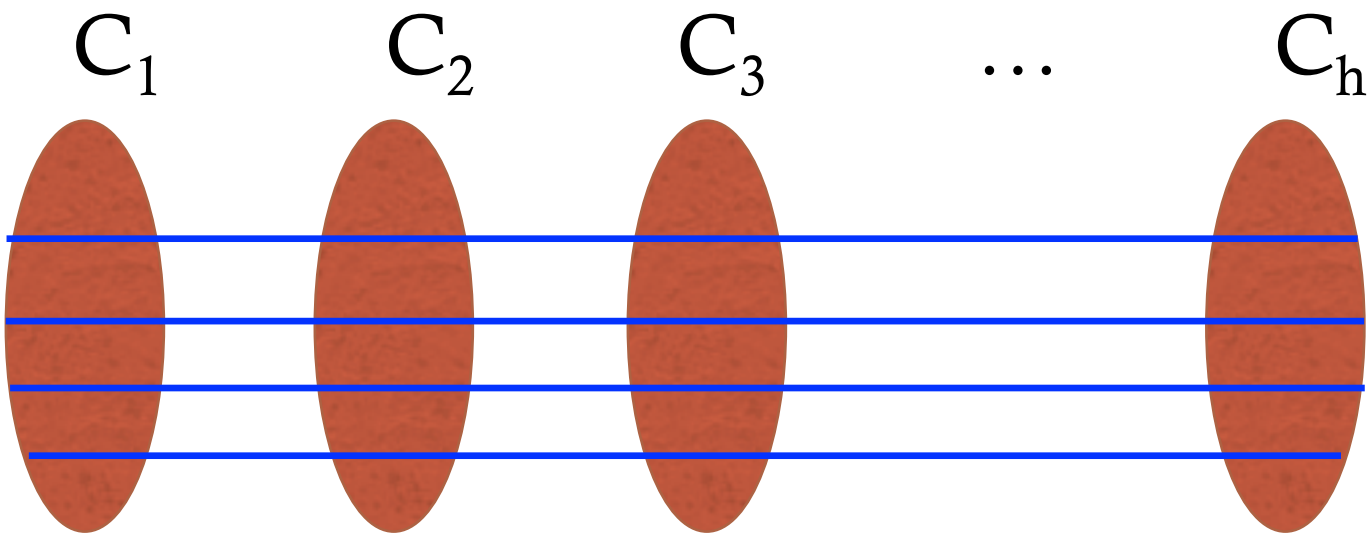

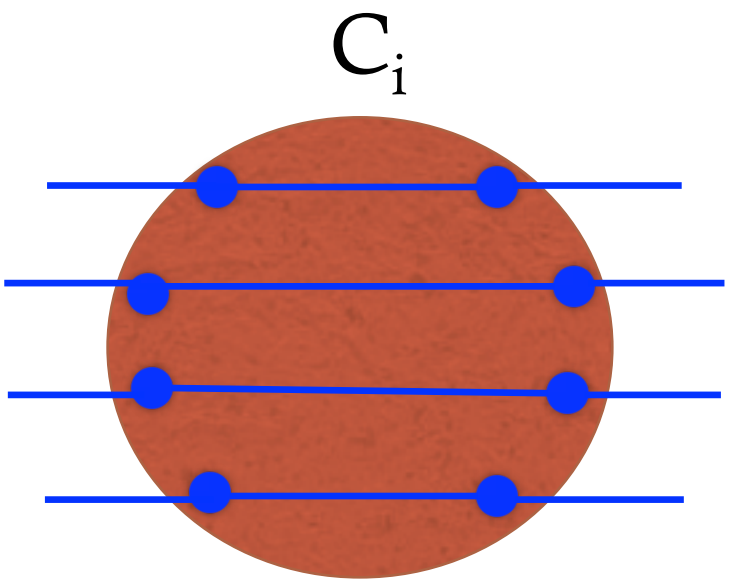

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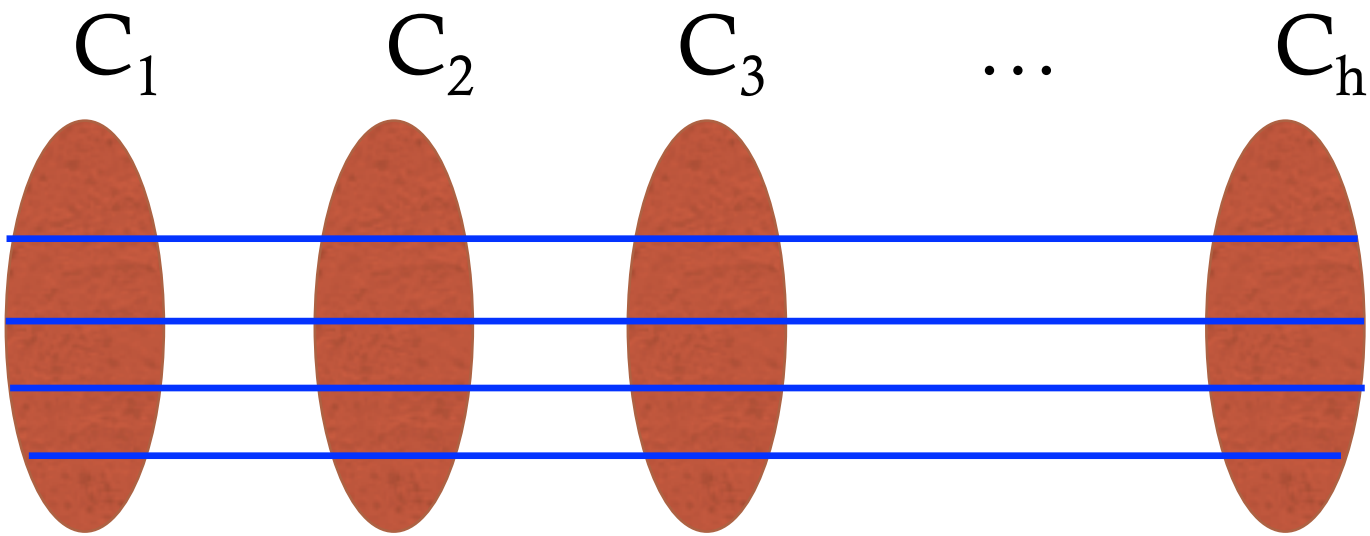

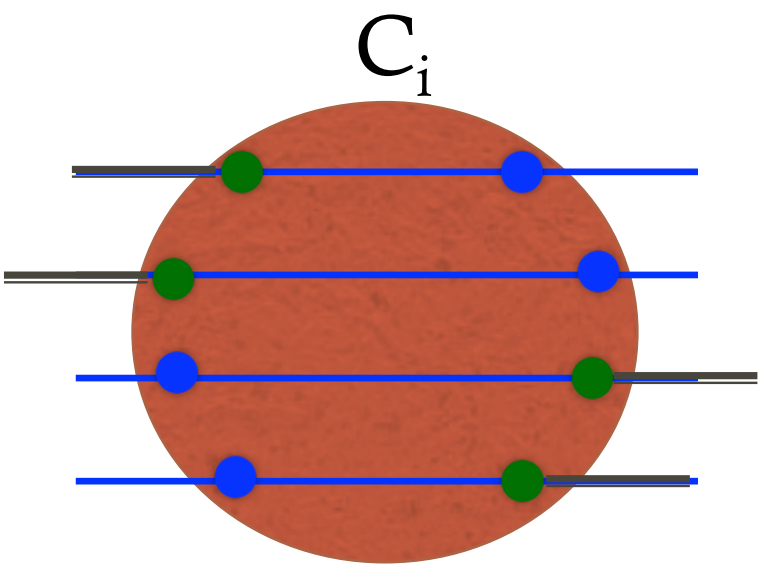

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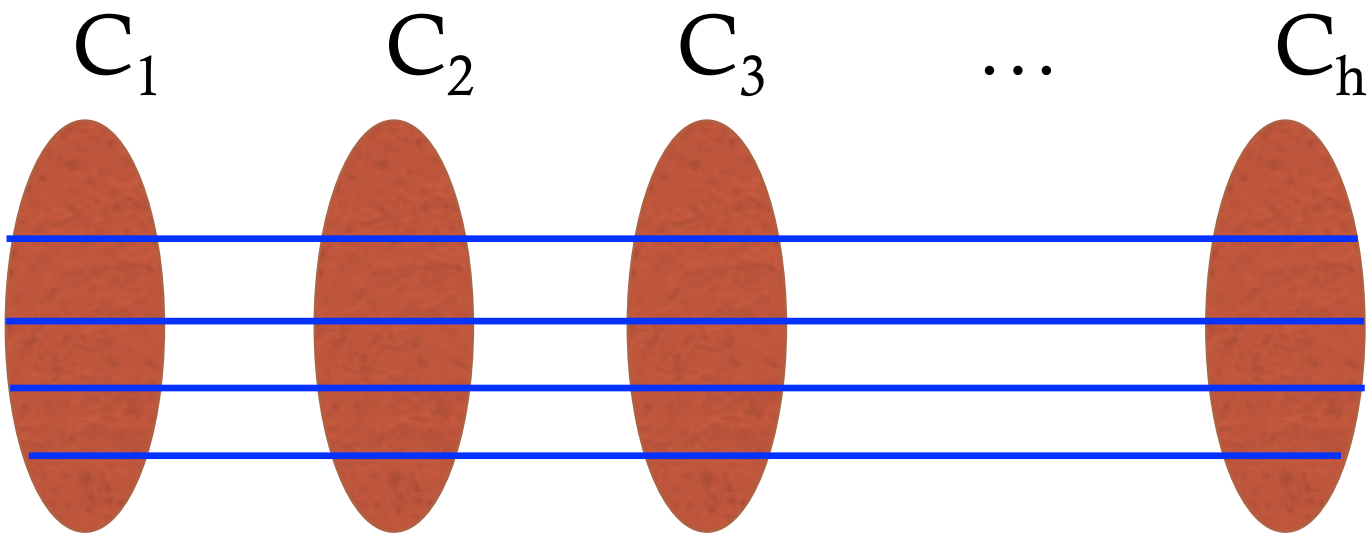

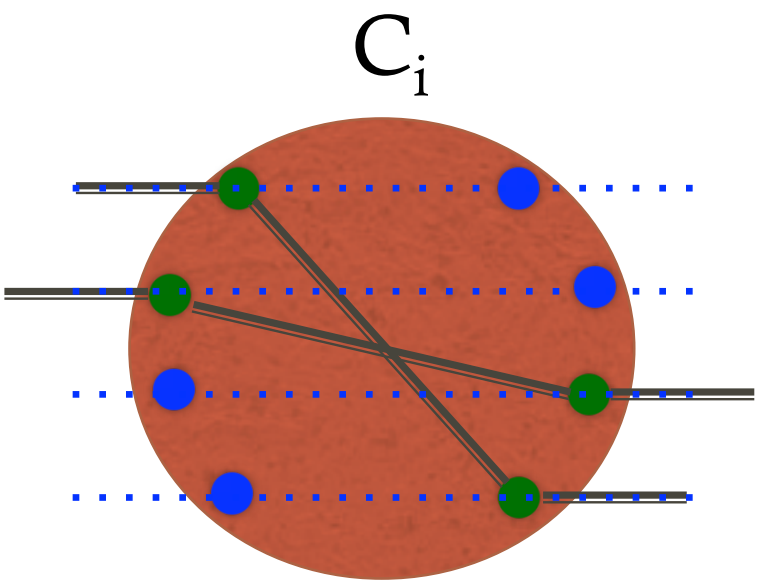

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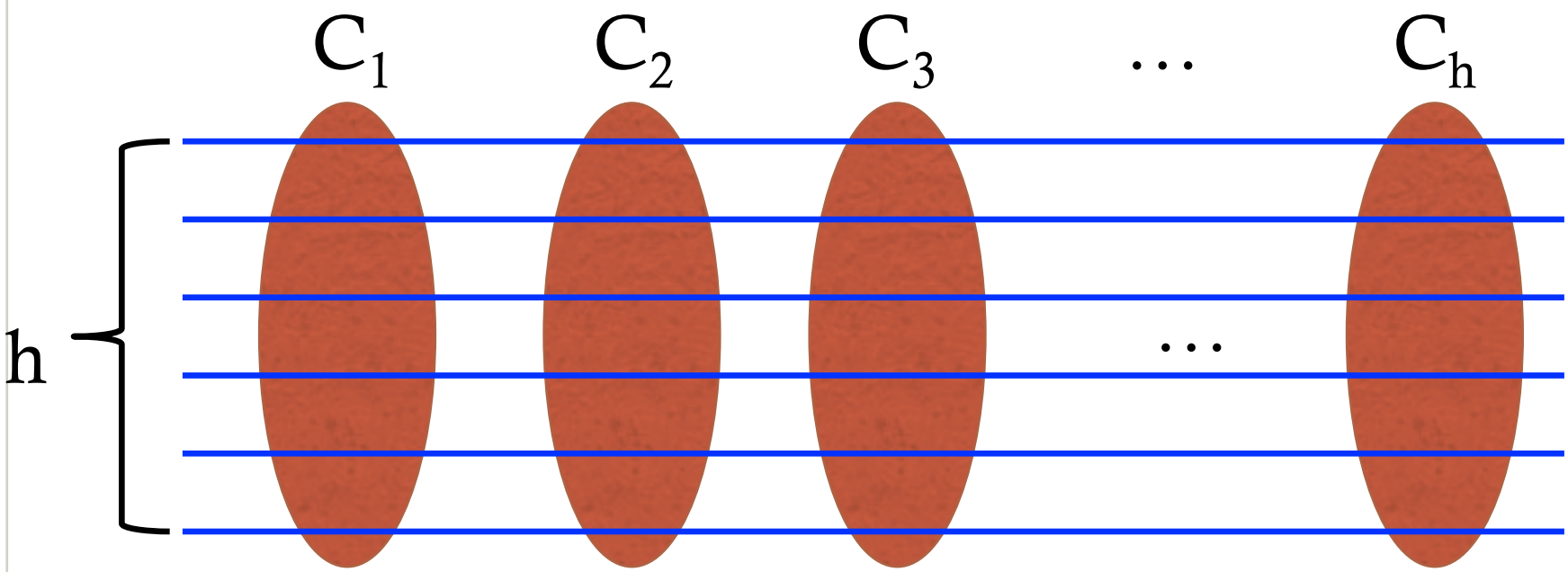


Theorem: Given a path-of-sets system, there is an efficient algorithm to find a grid minor of size $\Omega(\sqrt{h})$

Corollary: enough to find a path-of-sets system with $h=\text{poly}(k)$, where k is the treewidth.

Relation to (h,r)-grill

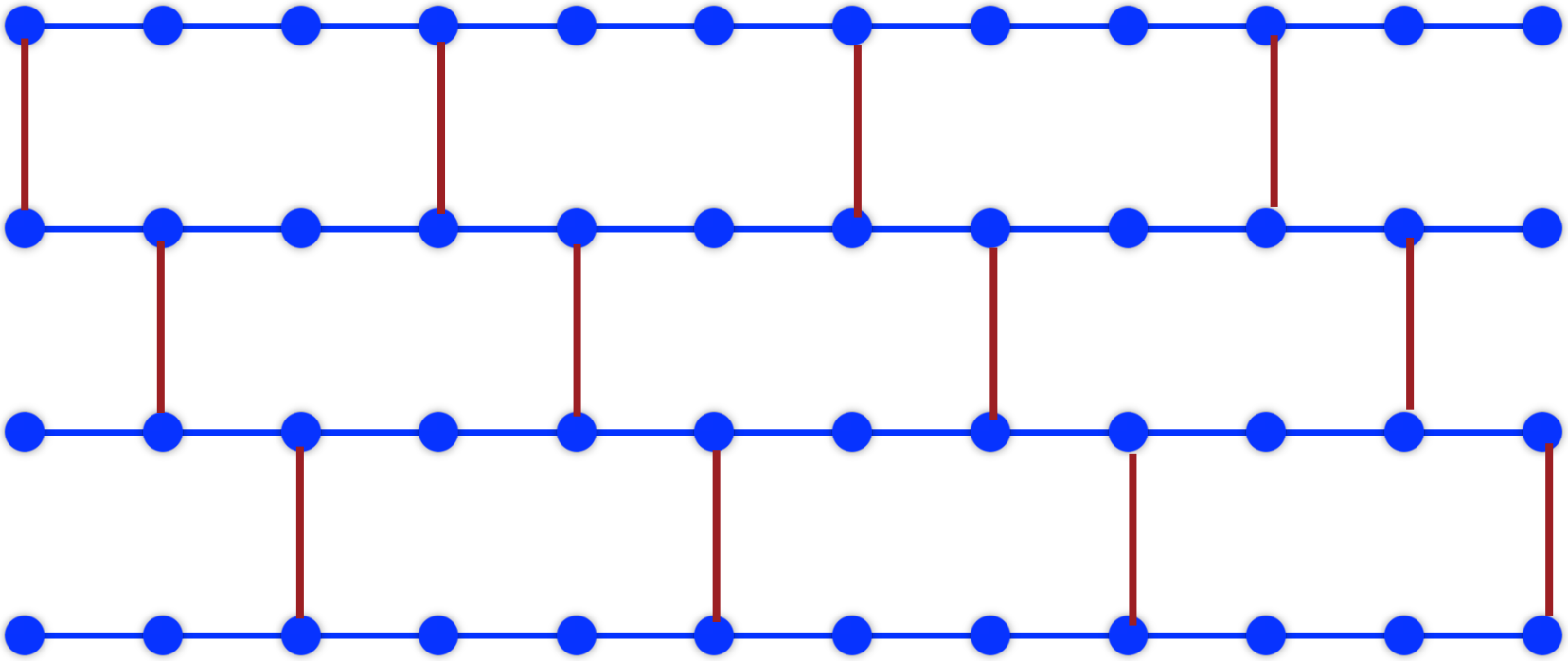
Thm [Leaf, Seymour '12]: Given a (h,r)-grill efficient algorithm to find a grid-minor.

(h,r) grills very closely related to (h,r)-path-of-sets-systems

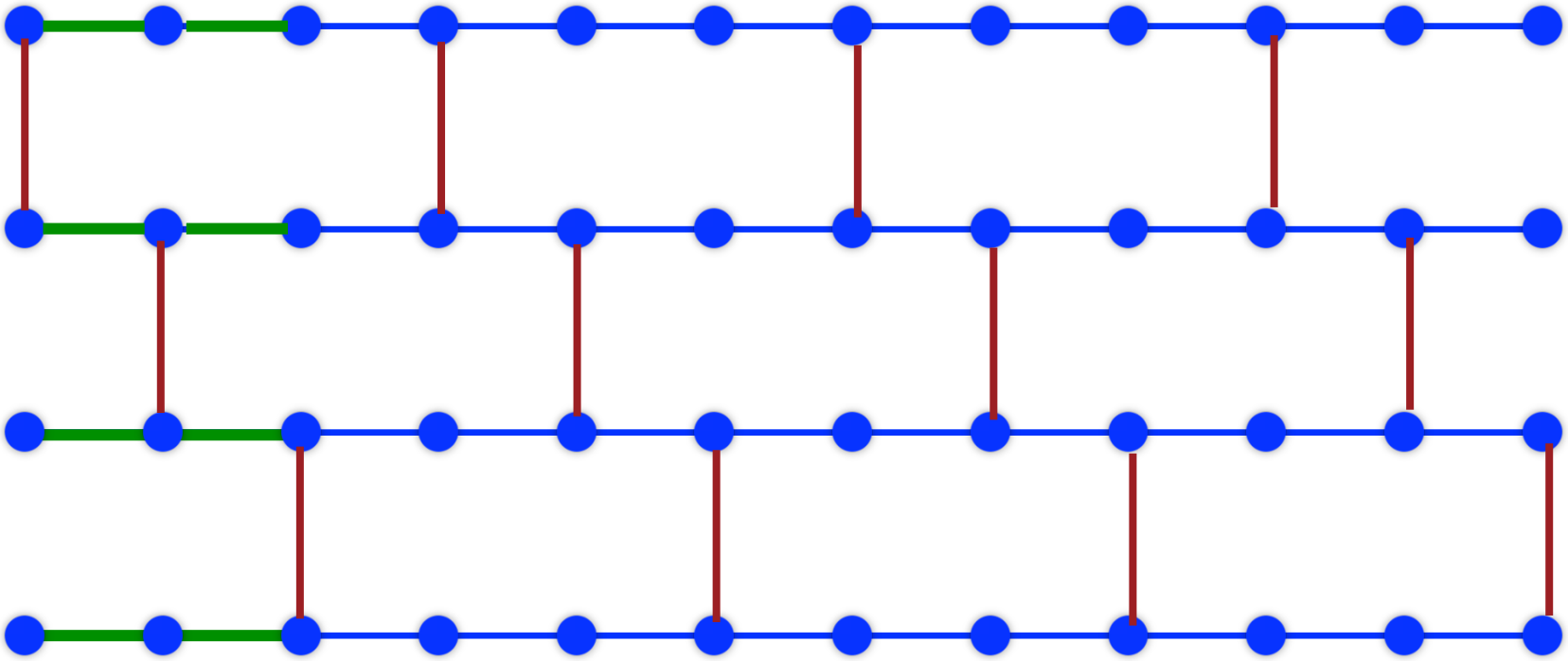
Our proof is essentially identical to theirs but path-of-sets-system gives slightly better parameters.

From Path-of-Sets System to Grid Minor

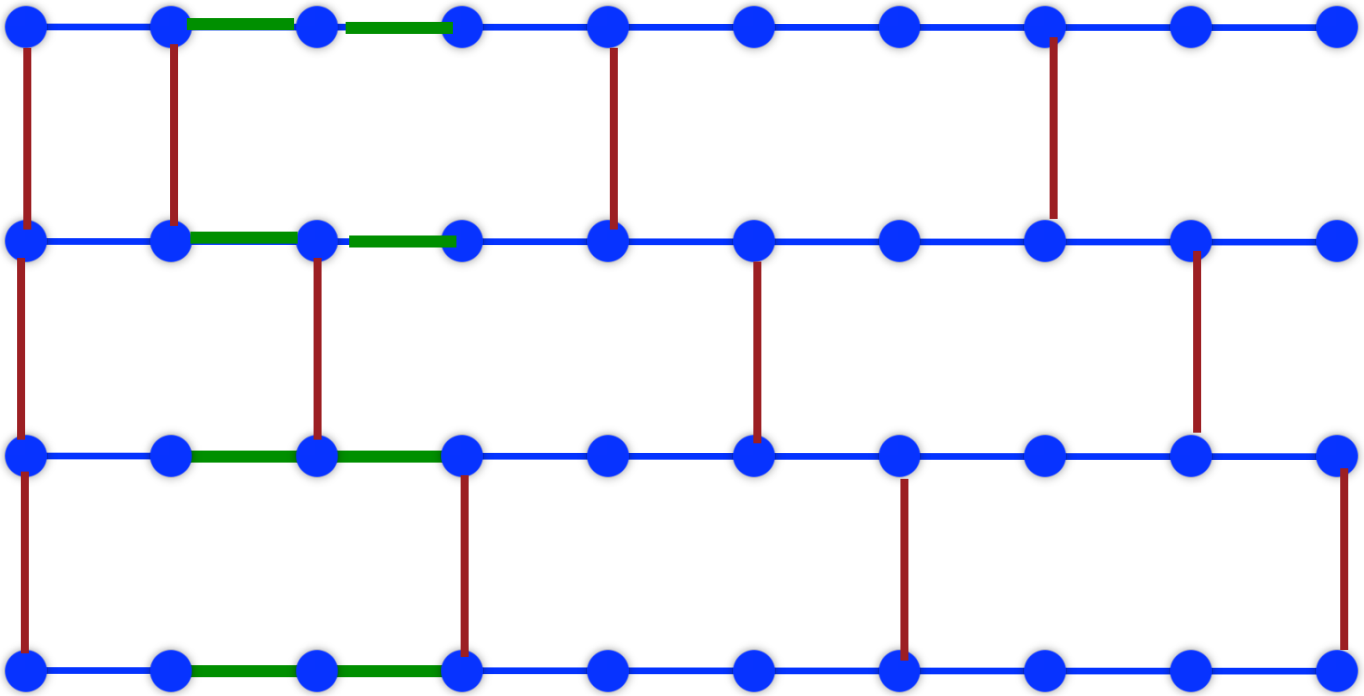
Building the Grid



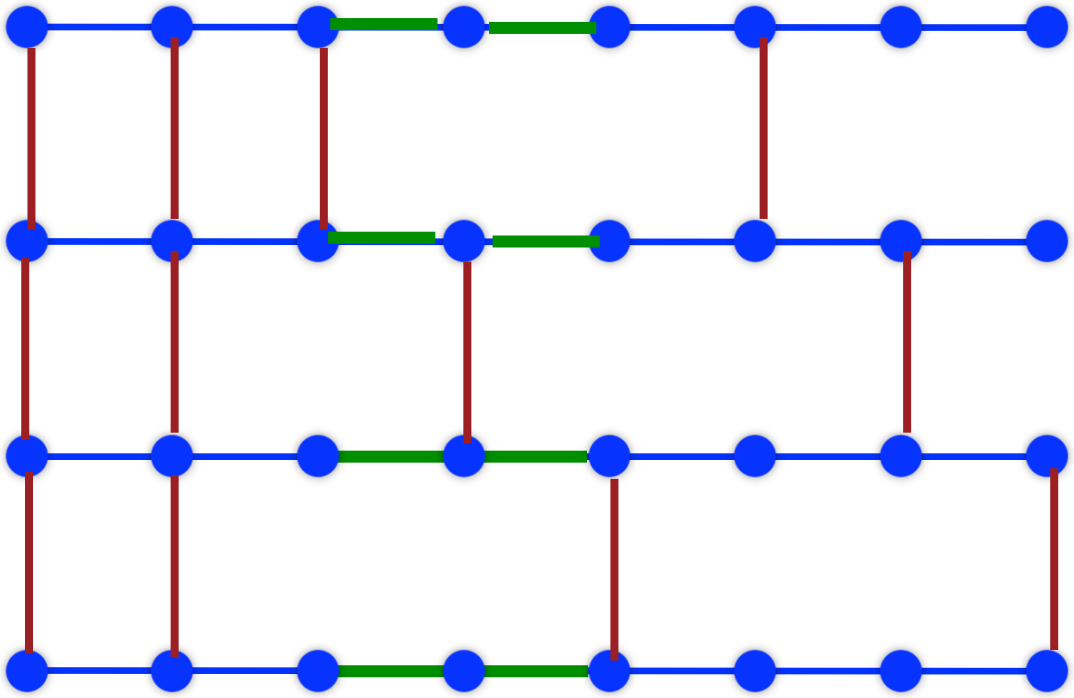
Building the Grid



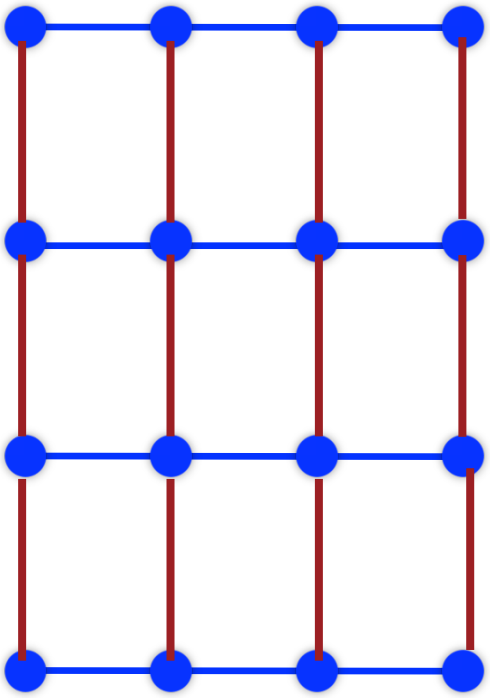
Building the Grid

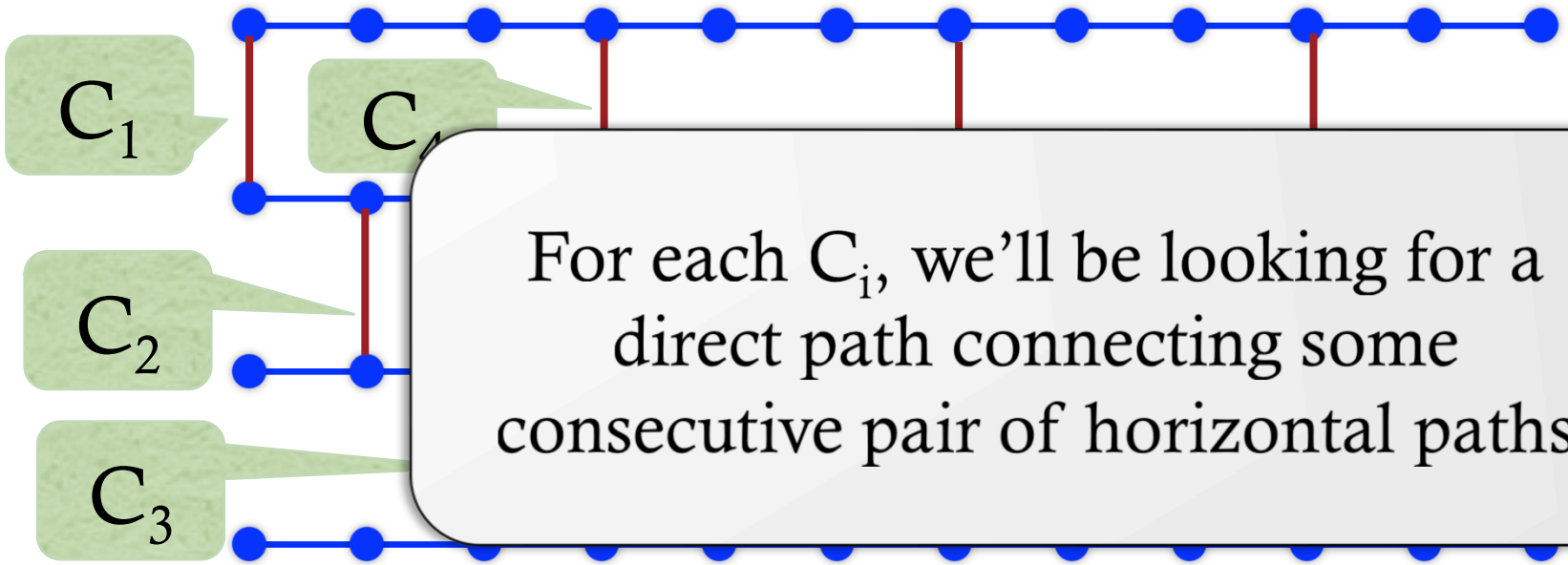


Building the Grid

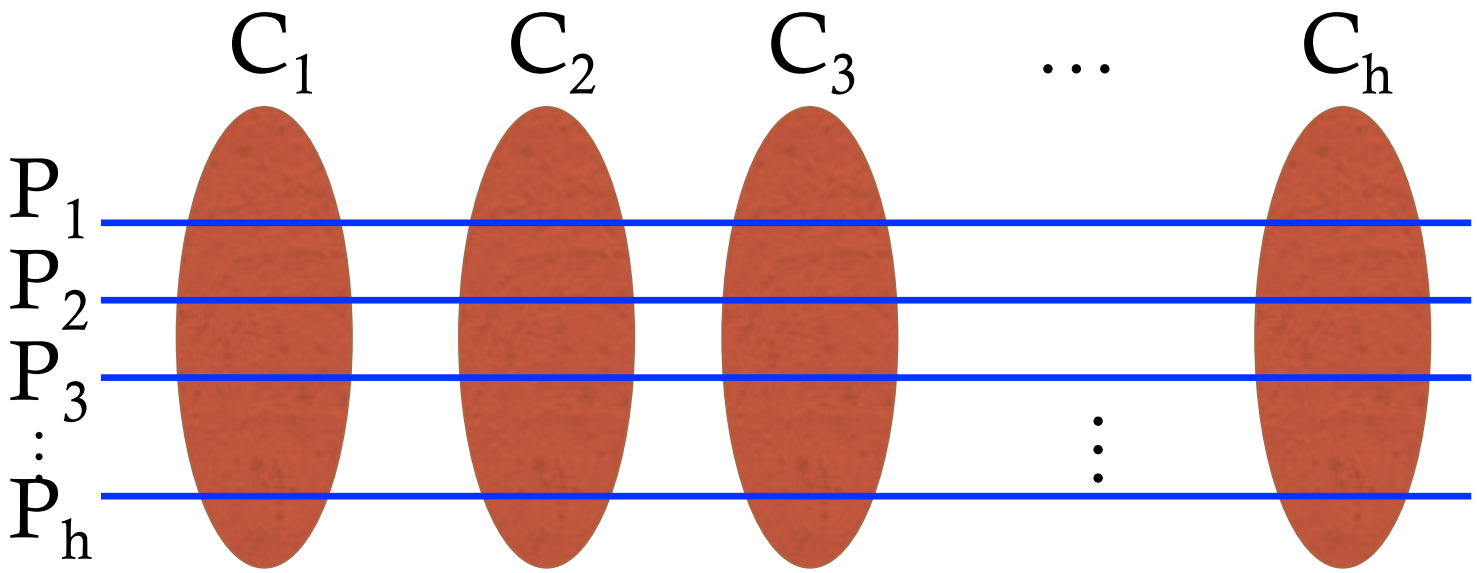


Building the Grid

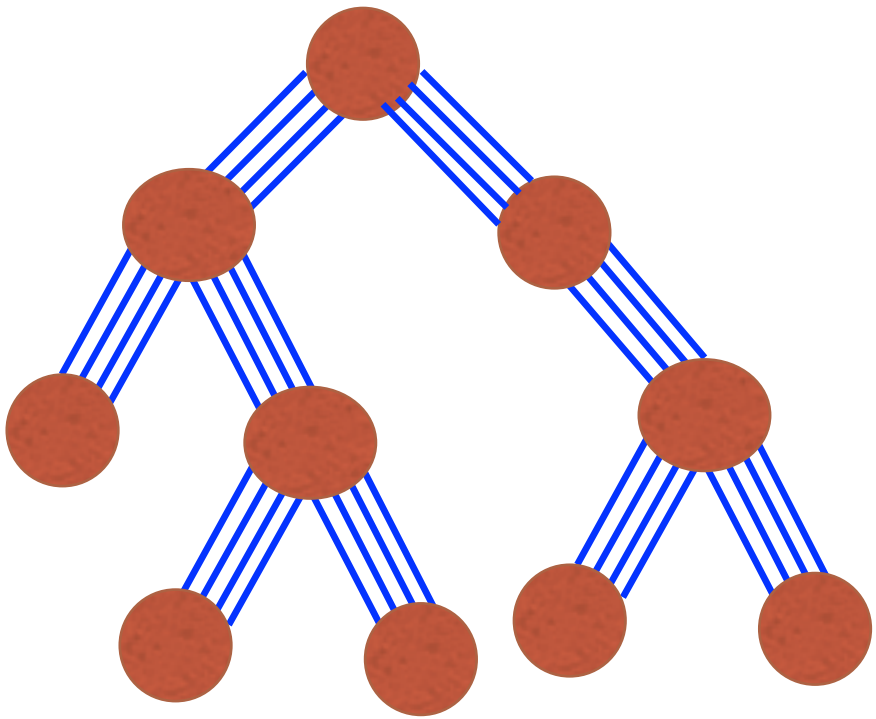


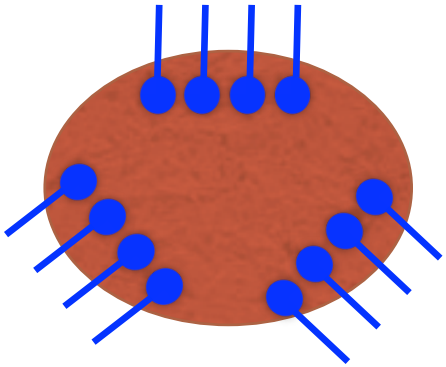


For each C_i, we'll be looking for a direct path connecting some consecutive pair of horizontal paths



Tree of Sets System

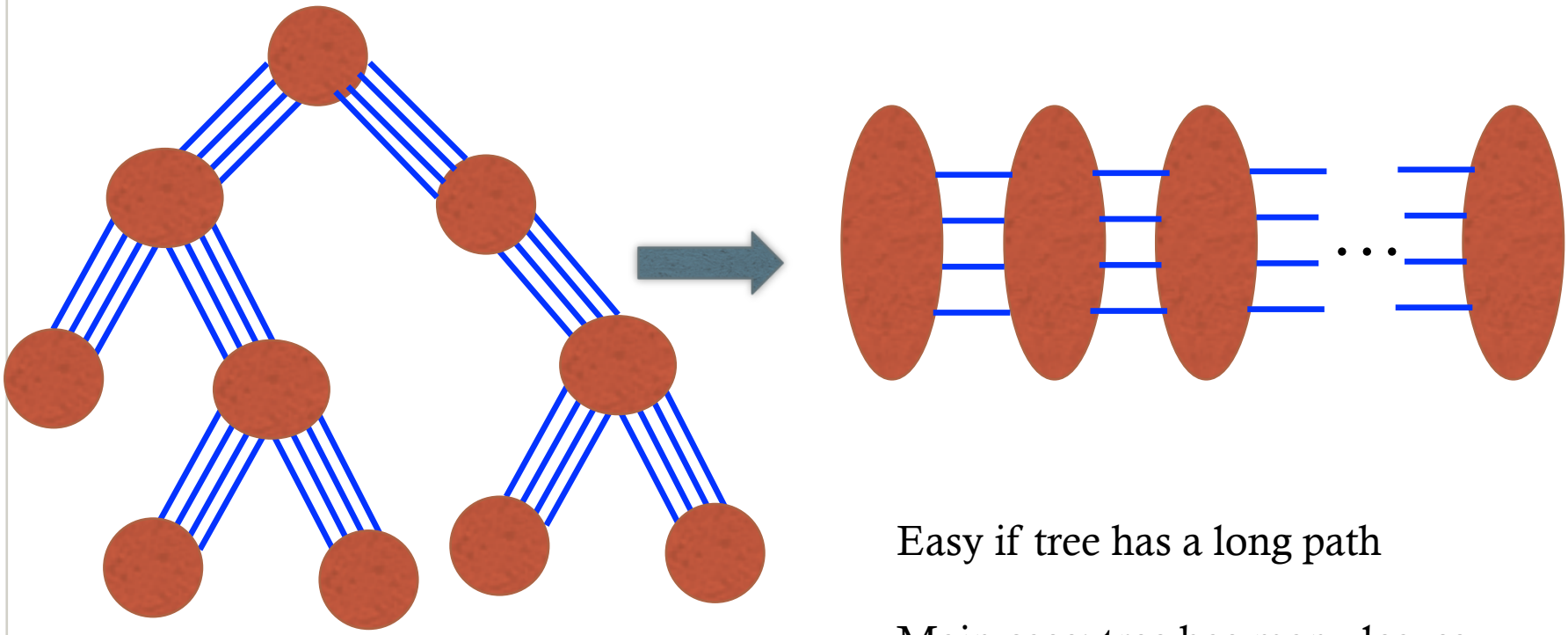




Interface is well-linked

deg 3 tree T on r nodes
h disjoint paths for each edge of T

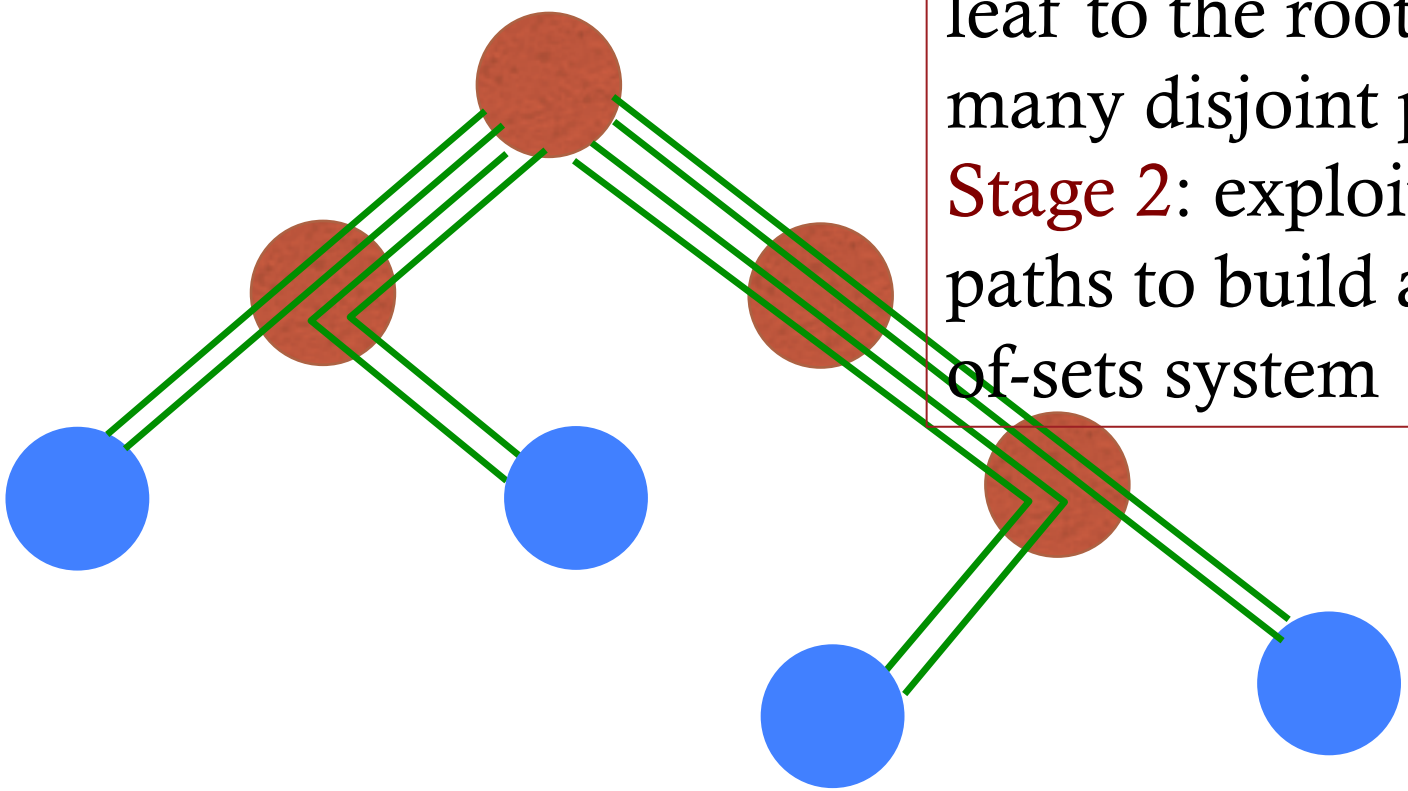
Tree to Path



Easy if tree has a long path

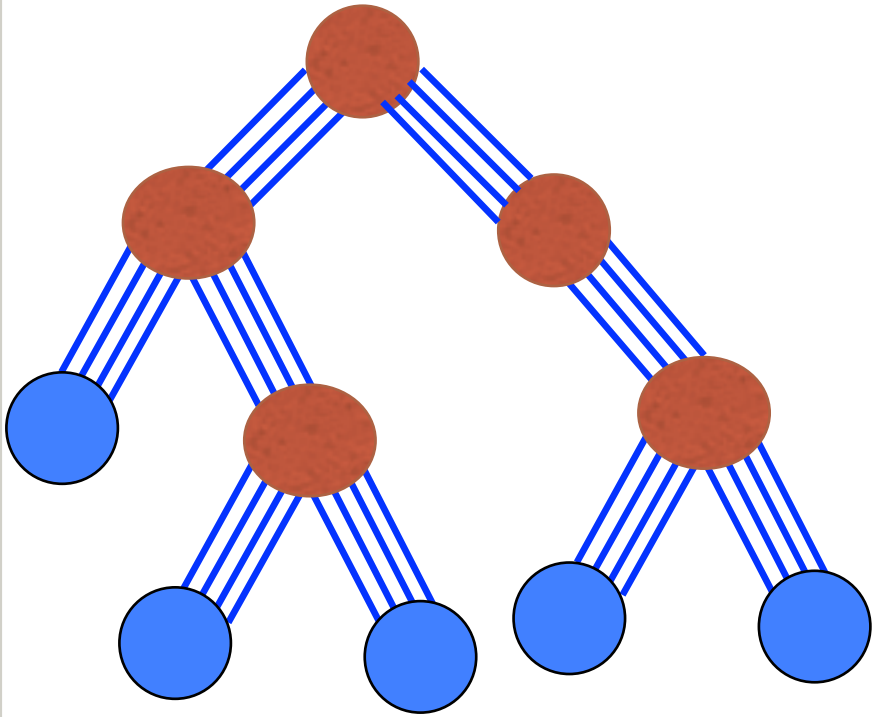
Main case: tree has many leaves

High-Level Idea

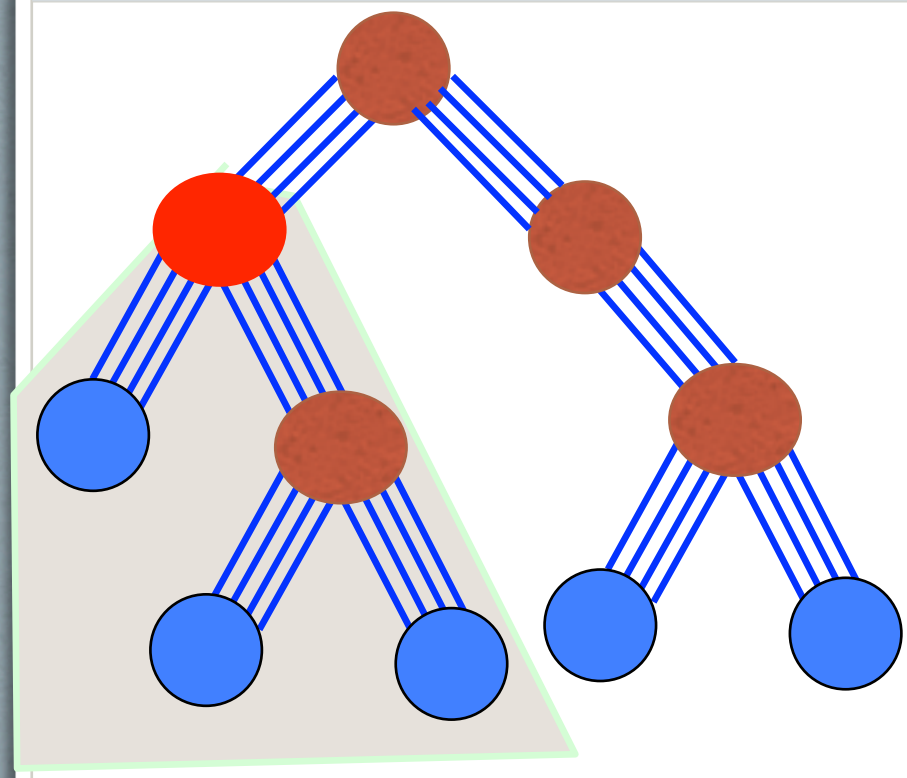


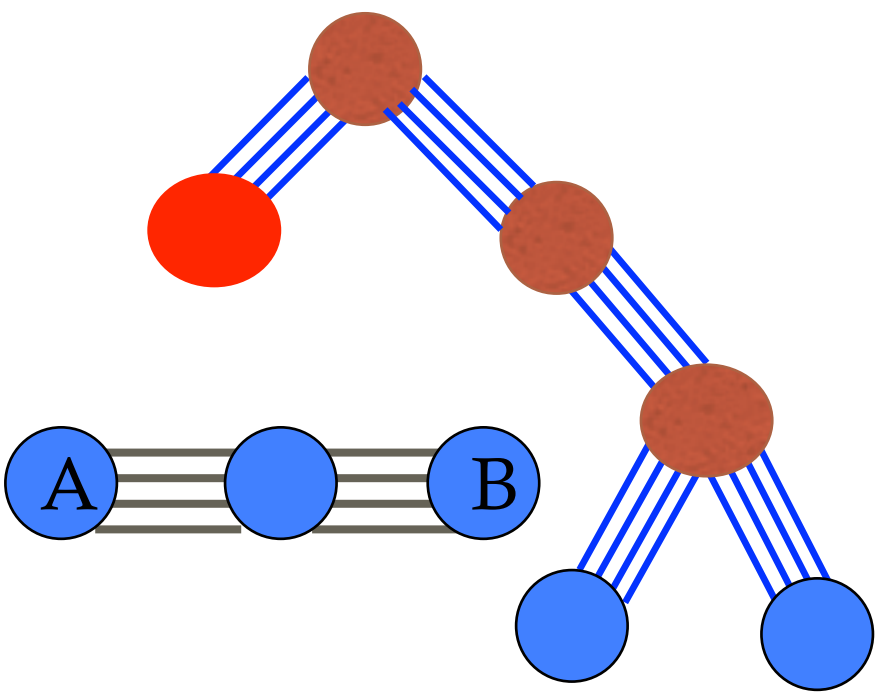
Stage 1: connect every leaf to the root by many disjoint paths
Stage 2: exploit these paths to build a path-of-sets system

Stage 2

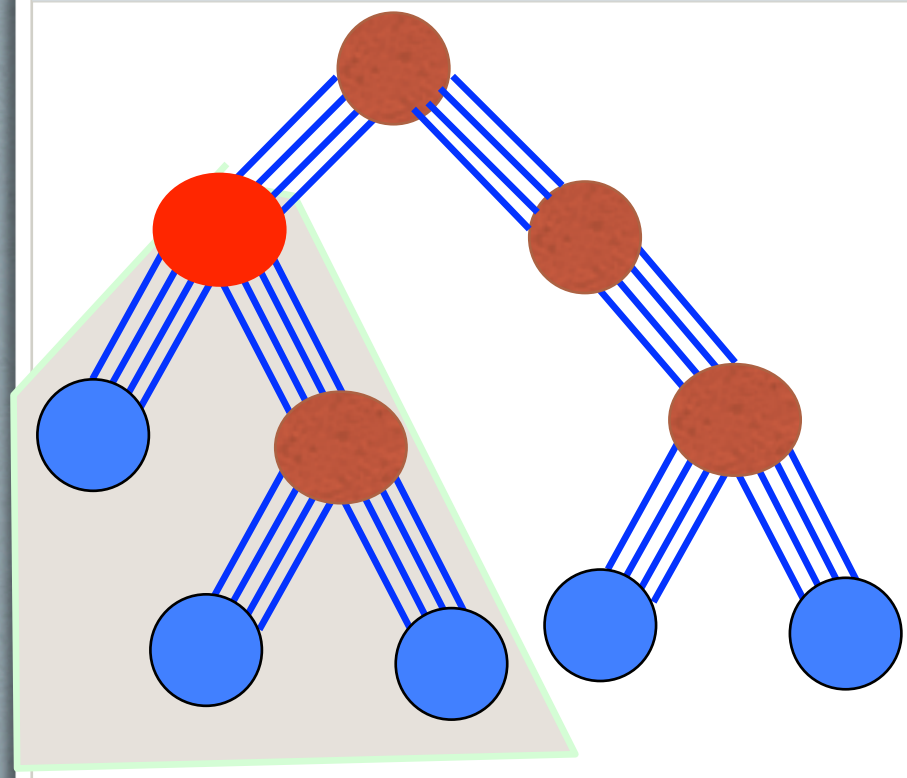


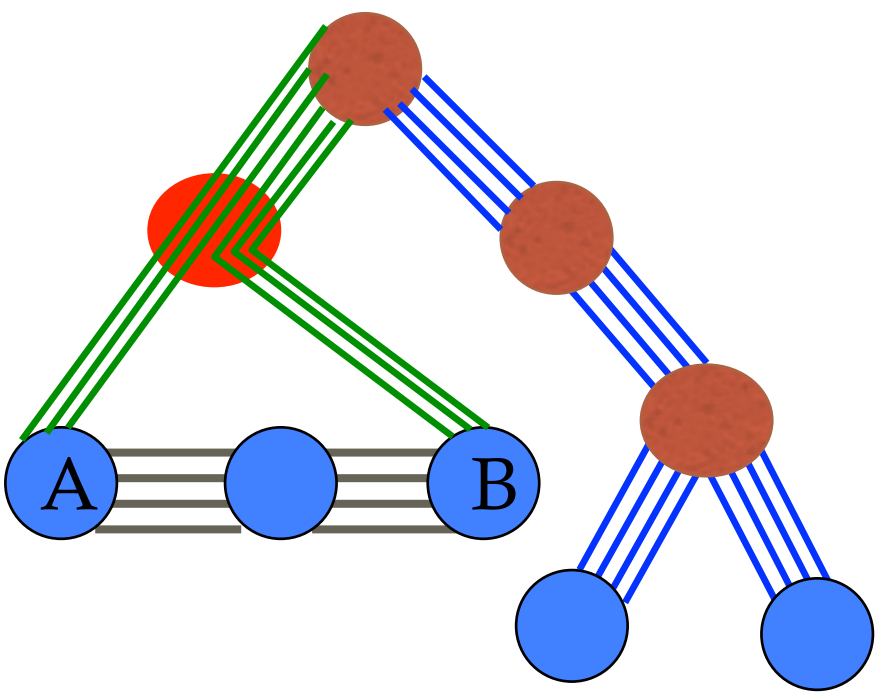
Stage 2



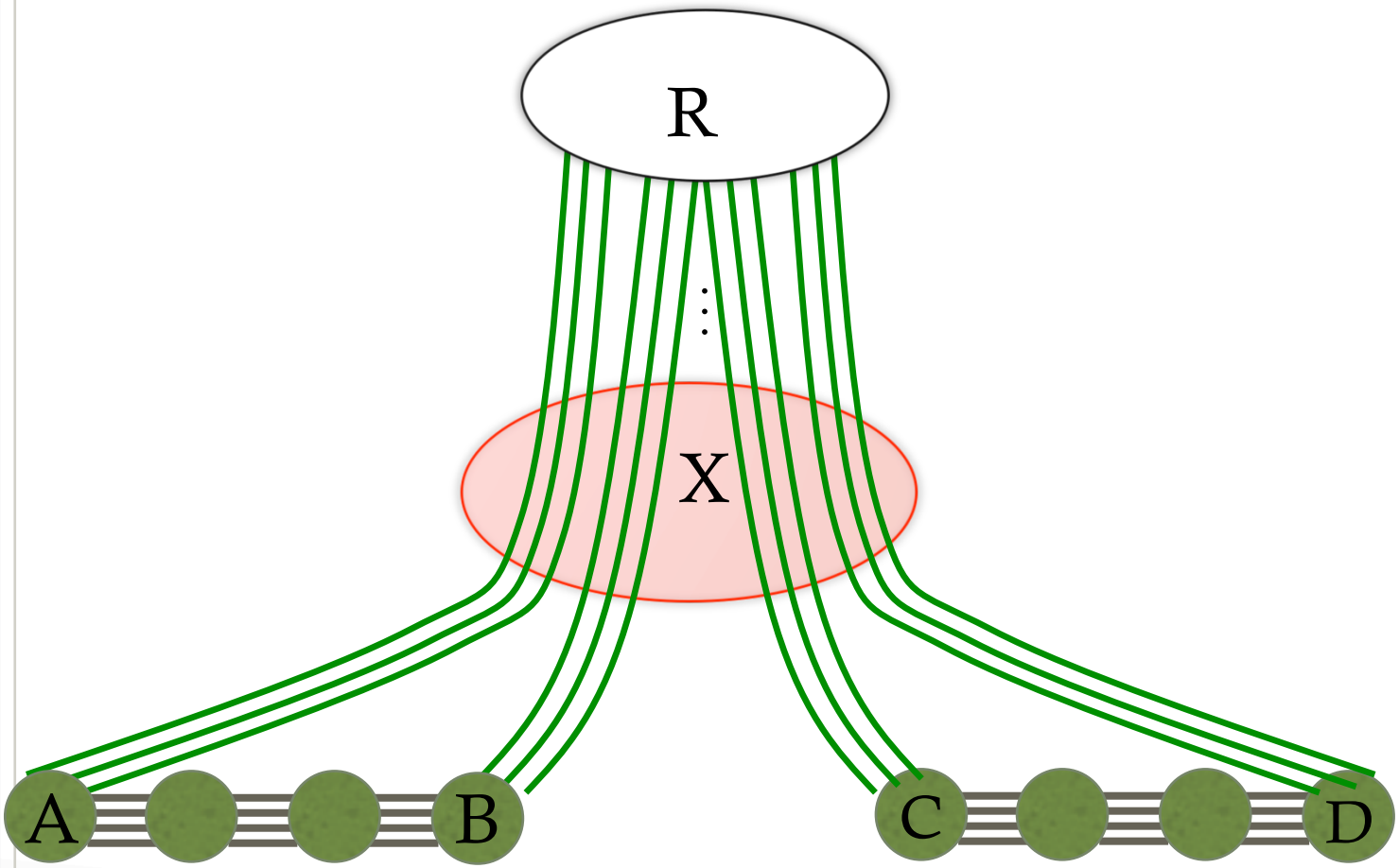


Stage 2

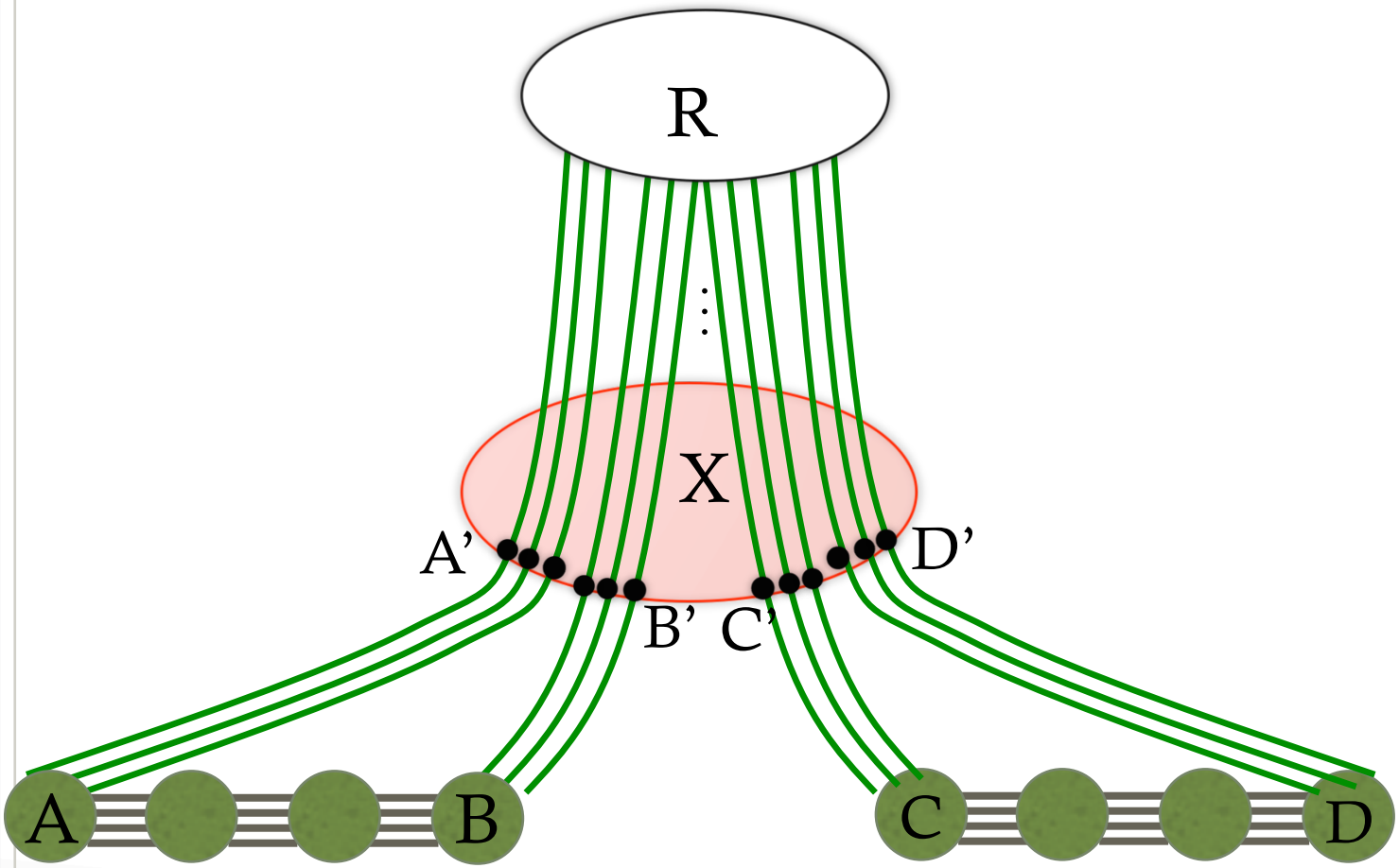




Stage 2



Stage 2



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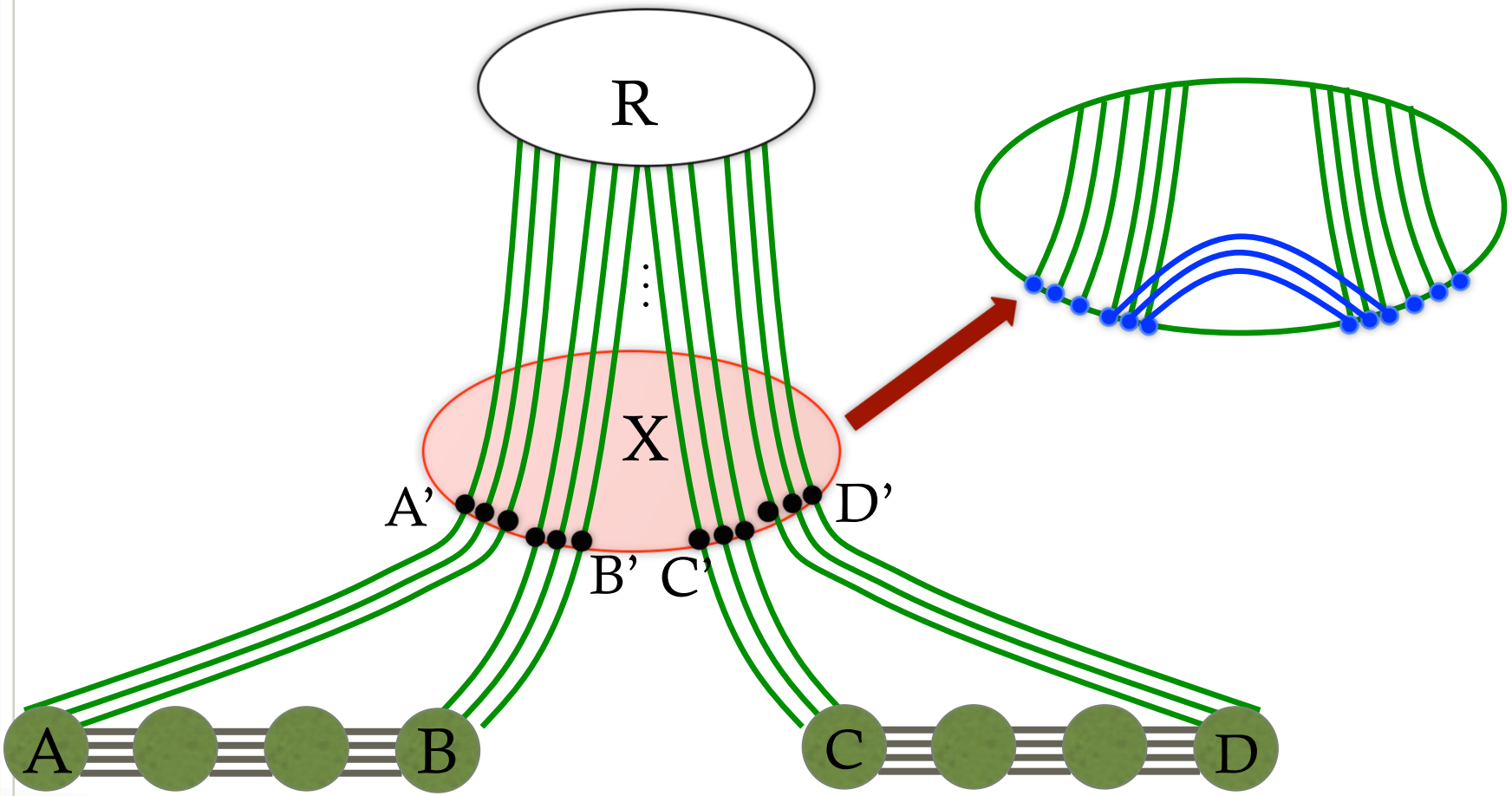

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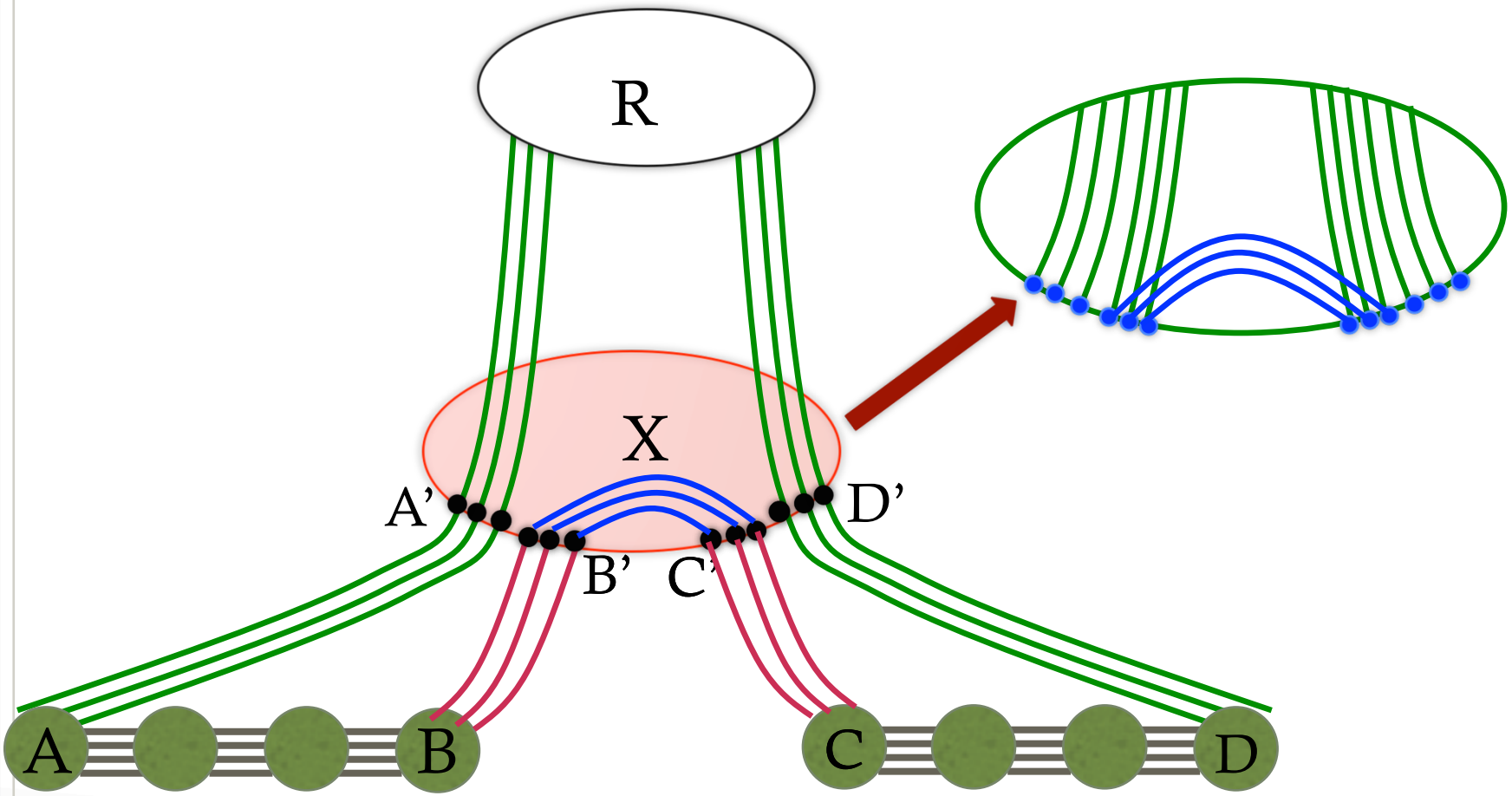
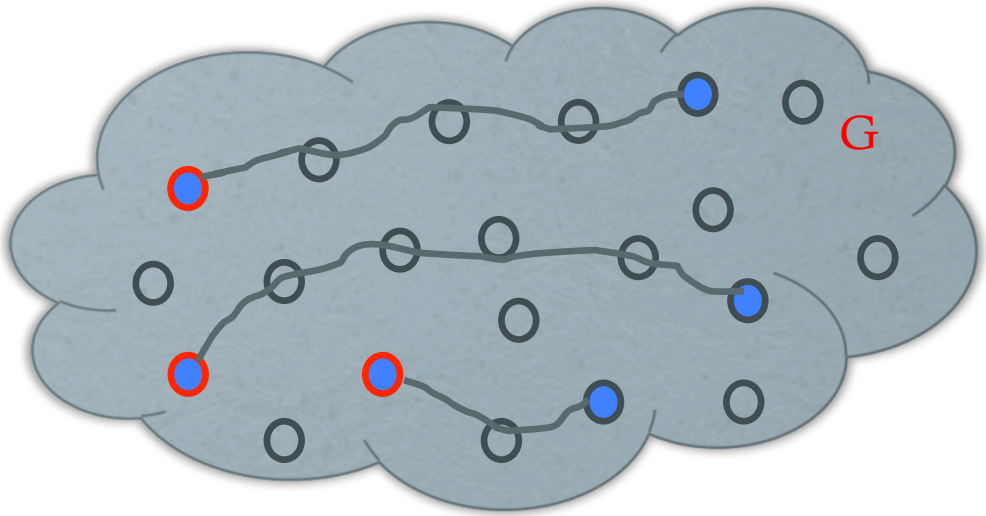


On Well-linked Sets

Well-linked Sets

A set $X \subseteq V$ is **well-linked** in G if for all $A, B \subseteq X$ there are $\min(|A|,|B|)$ node-disjoint A-B paths



$X \subseteq V$ is **edge-well-linked** if paths are edge-disjoint

Fractionally Well-linked Sets

For $\alpha \in (0,1]$ a set $X \subseteq V$ is α-**well-linked** in G if for all $A, B \subseteq X$ there are $\min(|A|,|B|)$ A-B paths with edge/node *congestion* $1/\alpha$

Grouping/Boosting Results on Well-linked Sets

[C-Khanna-Shepherd'05]

Given promise that $X \subseteq V$ is α-**edge-well-linked** in G can *efficiently* find $X' \subseteq X$ such that X' is **edge-well-linked** in G and

$$|X'| = \Omega(\alpha |X|)$$

Similar guarantee for node-well-linkedness but X' was only ¼-node-well-linked

Grouping/Boosting Results on Well-linked Sets

[C-Chuzhoy'13]

- $X \subseteq V$ is α-**edge-well-linked** in G
- X is partitioned into $X_1, X_2, \ldots, X_r$
- $\Delta(G)$ is maximum degree of G

Exists $X' \subseteq X$ such that

- X' is **node-well-linked** in G
- for $1 \leq j \leq r$, $|X' \cap X_j| = \Omega(\alpha |X_j| / \Delta^2(G))$

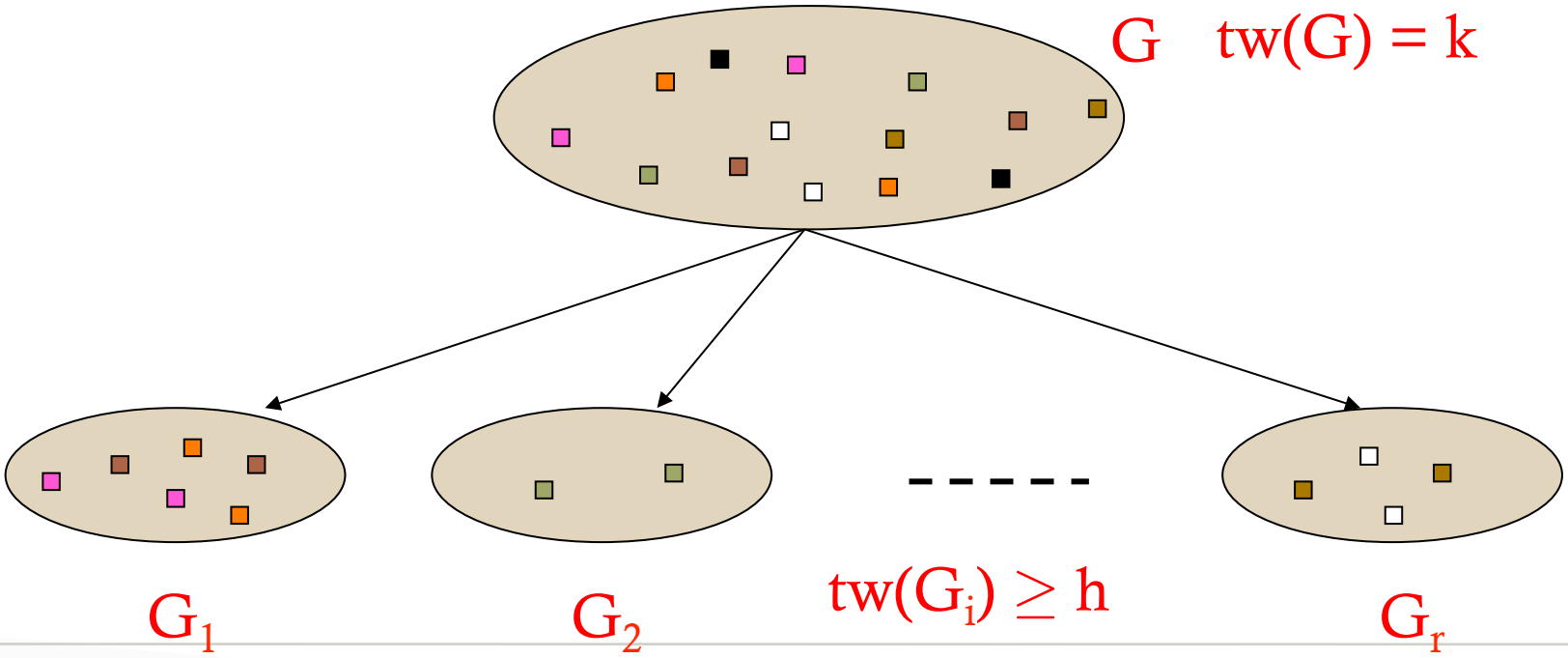
Efficiency results in extra log factor loss.

Decomposition and Tree of Sets

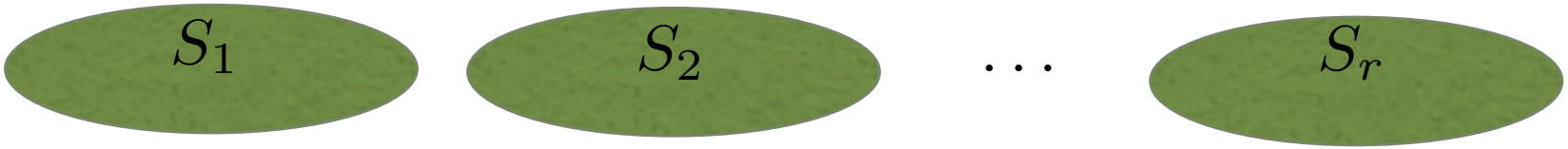
Mostly from [Chuzhoy'11,Chuzhoy-Li'12] + node-connectivity ideas from [C-Ene'13]

Treewidth Decomposition

- $r\,h^2 \leq k/\mathrm{polylog}(k)$ or
- $r^3\,h \leq k/\mathrm{polylog}(k)$



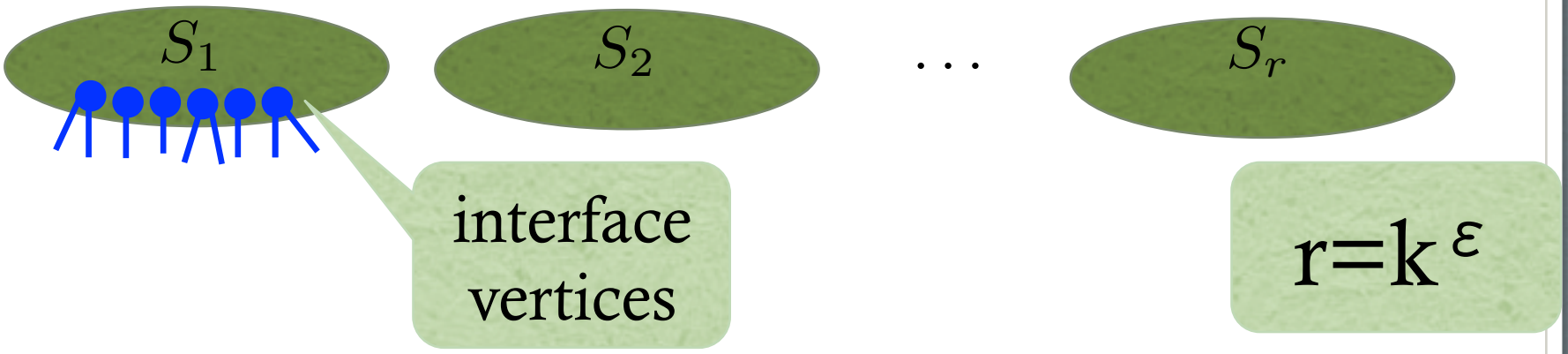
A Family of Good Routers



$r = k^{\varepsilon}$

- r disjoint clusters $S_1, \ldots, S_r$
- For each S_i, the interface vertices are $1/r^2$-well-linked *inside* S_i

A Family of Good Routers



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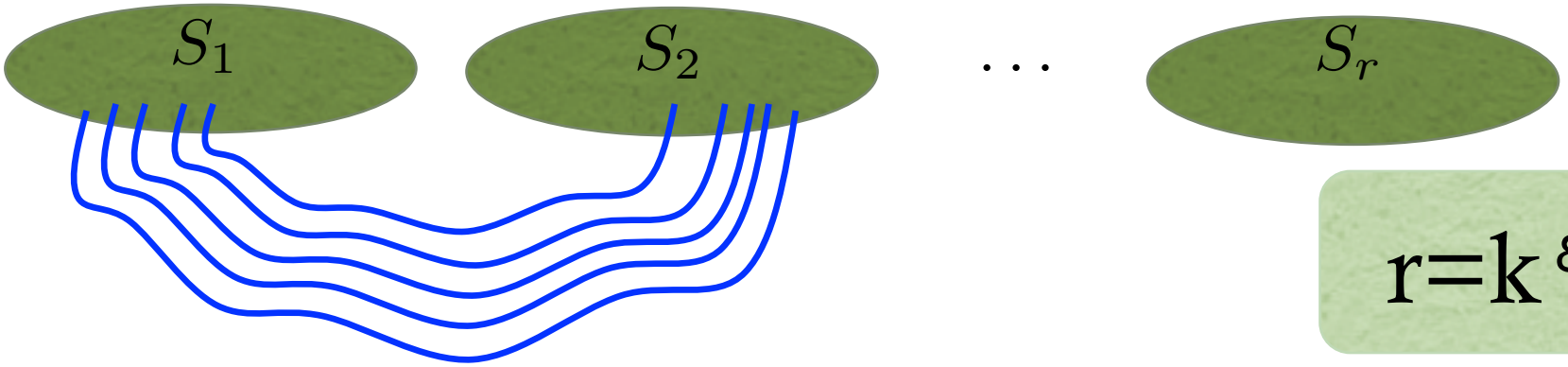
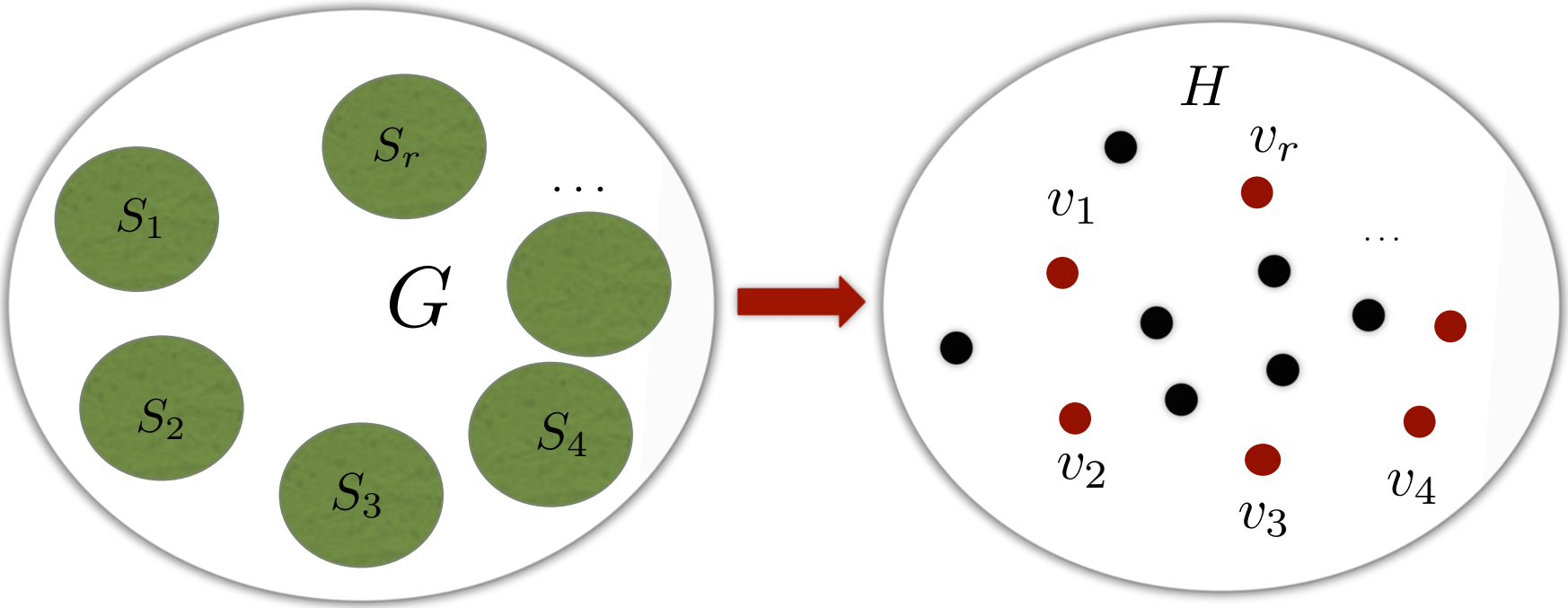
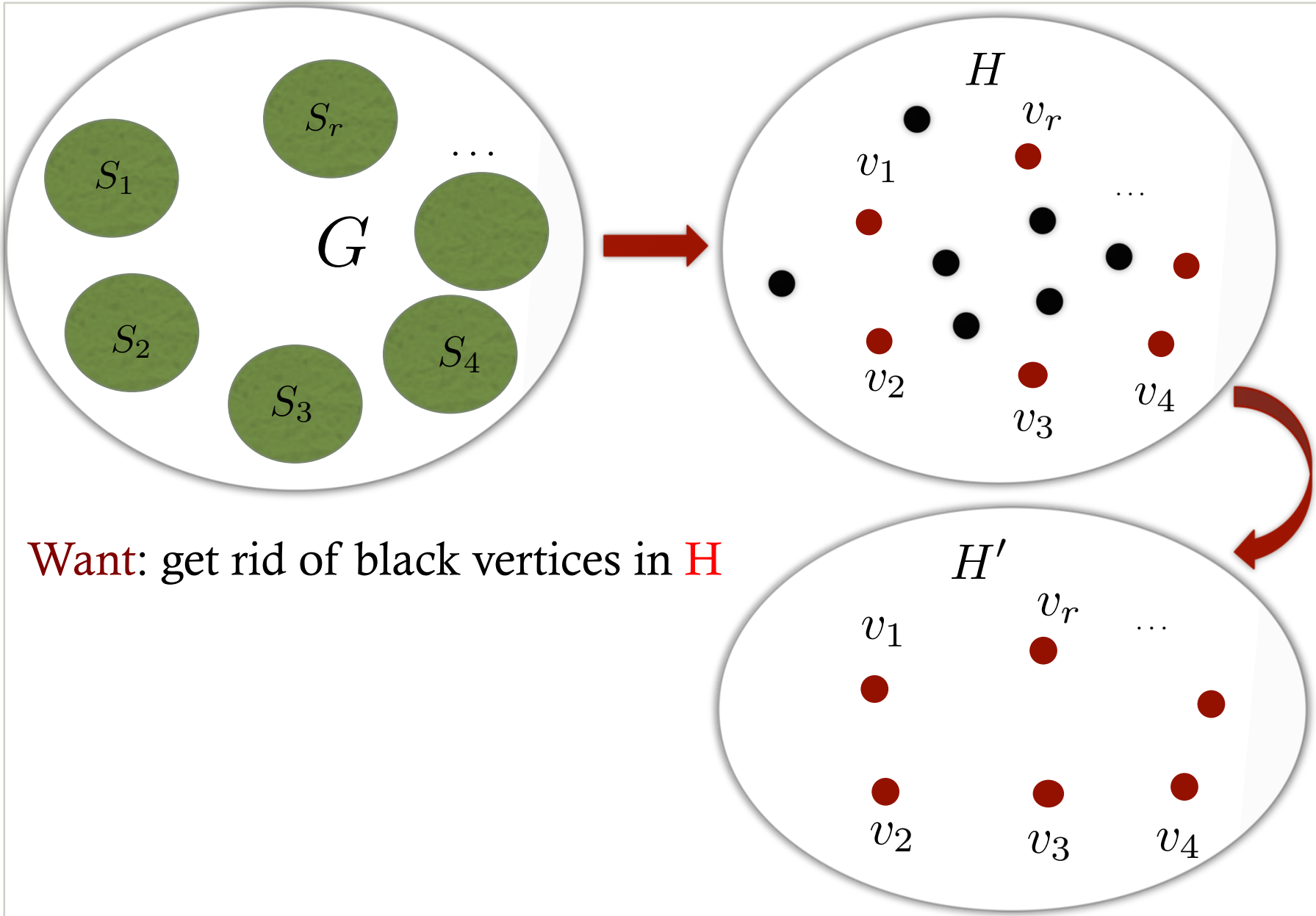


$r=k^{\varepsilon}$

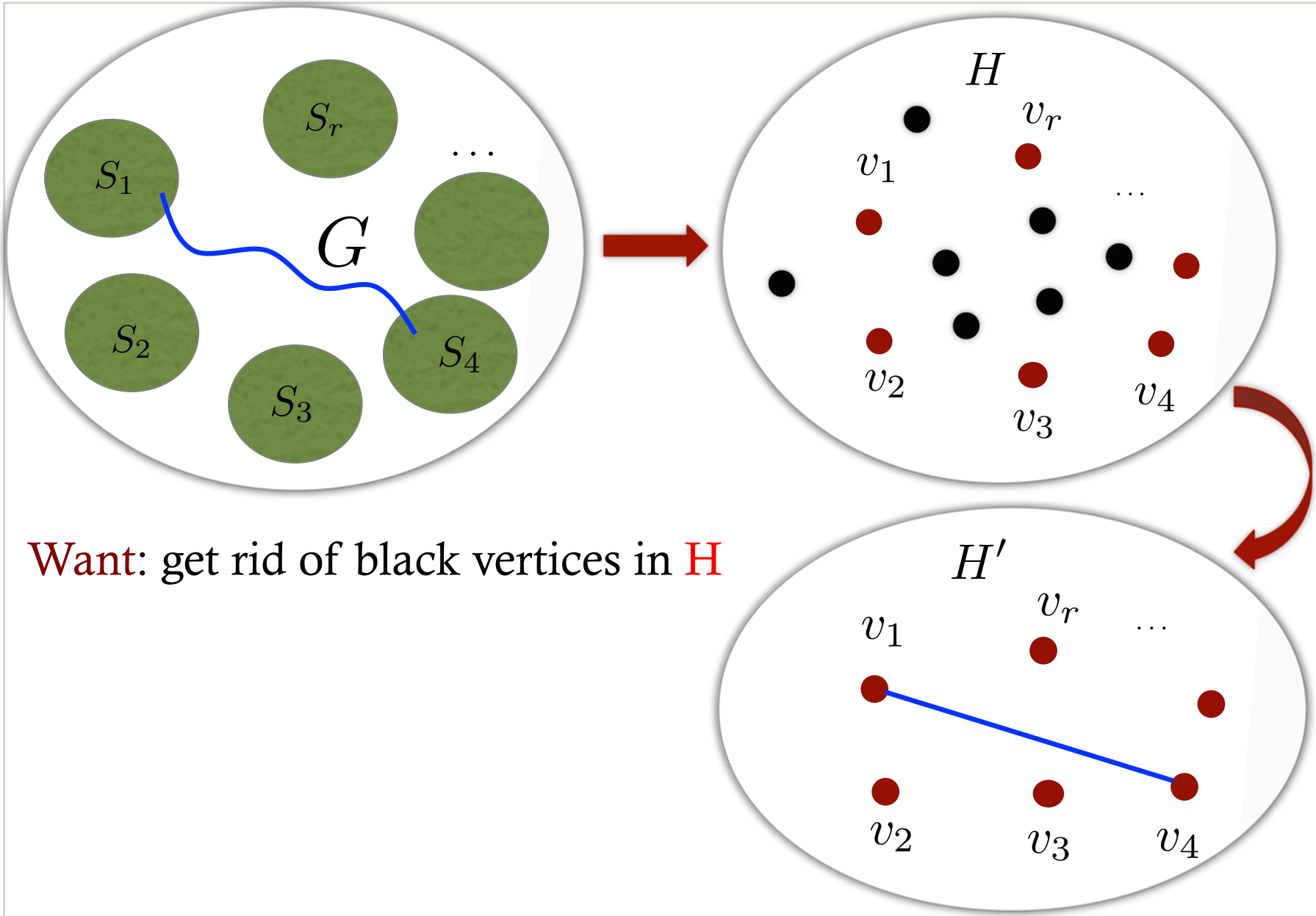
- r disjoint clusters $S_1,\ldots,S_r$
- For each S_i, the interface vertices are $1/r^2$-well-linked inside S_i
- For each i, j, there are $p=r^{20}$ node-disjoint paths connecting S_i to S_j. They are linked.

From Good Family to Tree of Set System





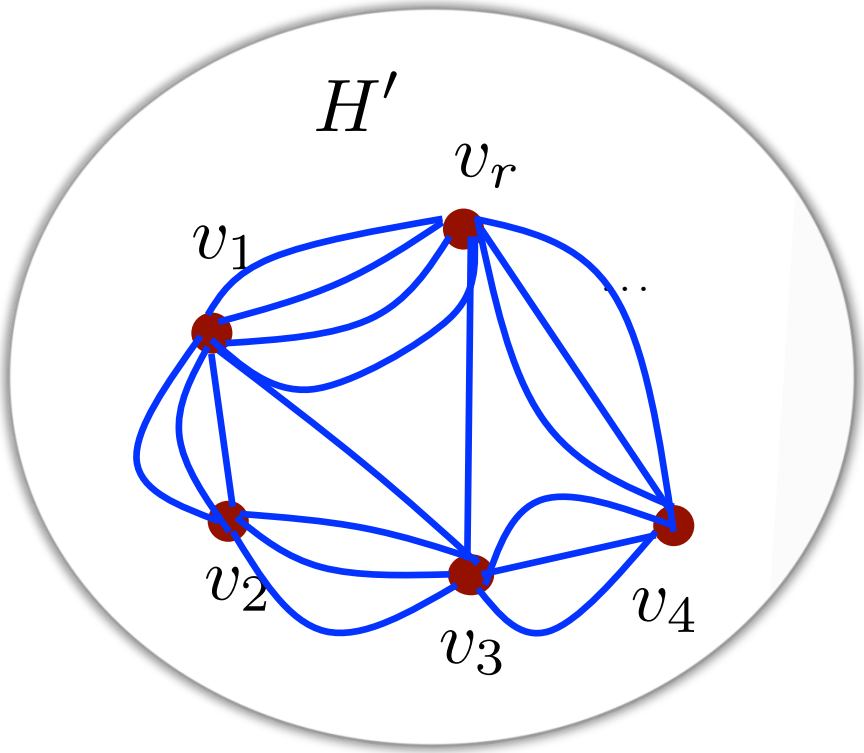
Want: get rid of black vertices in H



Want: get rid of black vertices in H

Graph H'

- edge in H' ➔ direct path in G;
- degree of every vertex is p/r
- every pair of vertices has p/r node-disjoint paths connecting them in H'

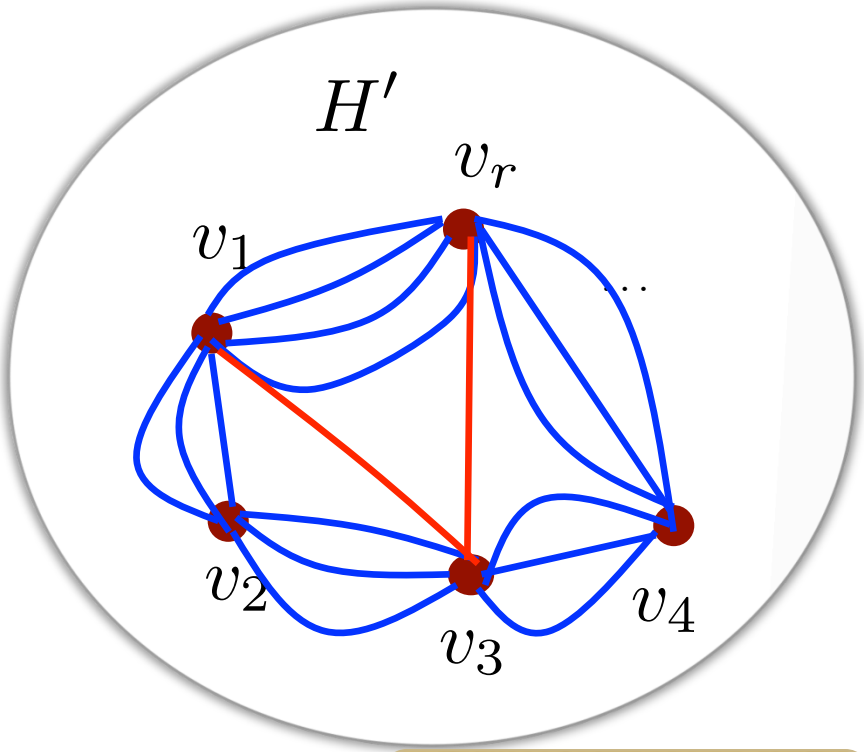


$$p = \Omega(r^{20})$$

Use edge/element connectivity graph reduction steps [Mader,Hind-Oellerman,C-Korula]

Graph H'

- edge in H' ➔ direct path in G;
- degree of every vertex is p/r
- every pair of vertices has p/r edge-disjoint paths connecting them in H'

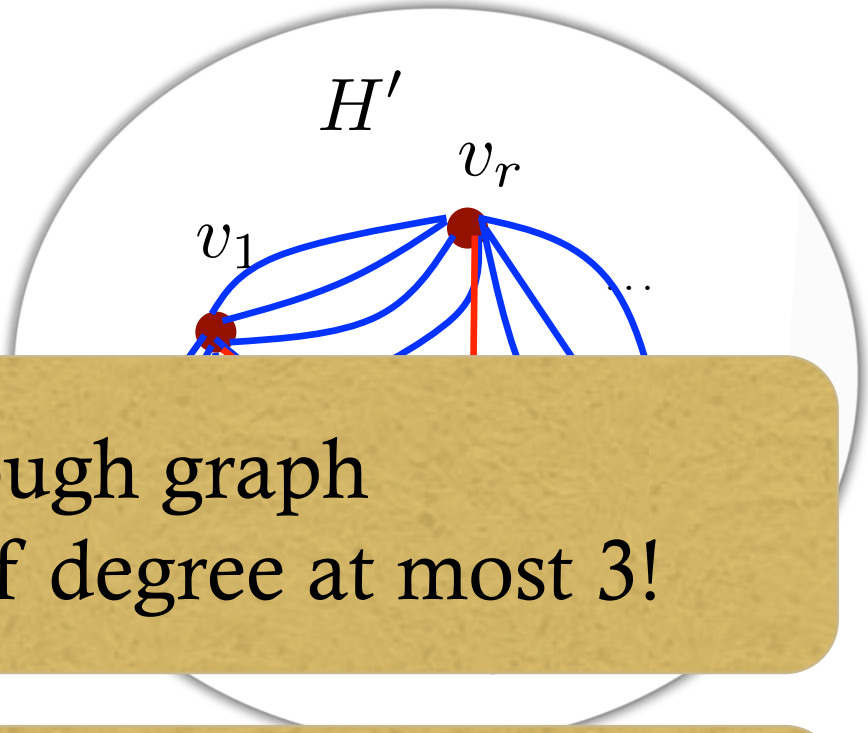


$$p = \Omega(r^{20})$$

If v_i,v_j are connected by fewer than p/r^4 edges, delete all edges connecting them.

Graph H'

- edge in H' ➔ direct path in G:



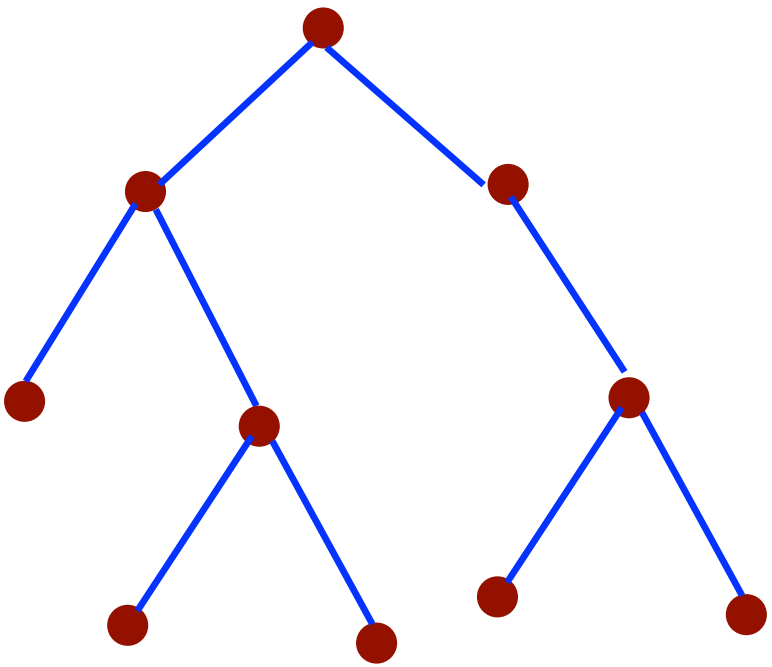
If H' is a 1-tough graph
it has spanning tree of degree at most 3!

p/r edge-disjoint paths

Need more work to obtain 1-tough graph by making degree uniform

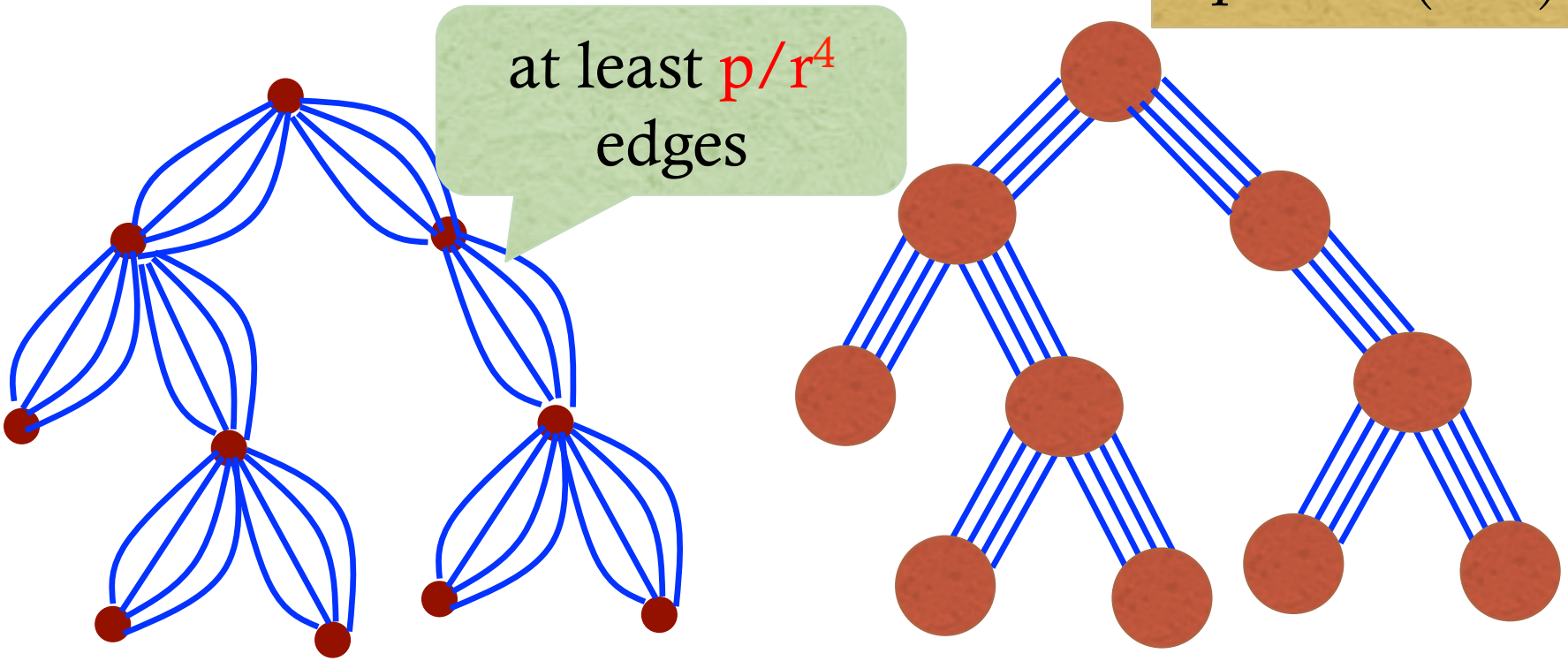
all edges connecting them.

Tree-of-Sets System



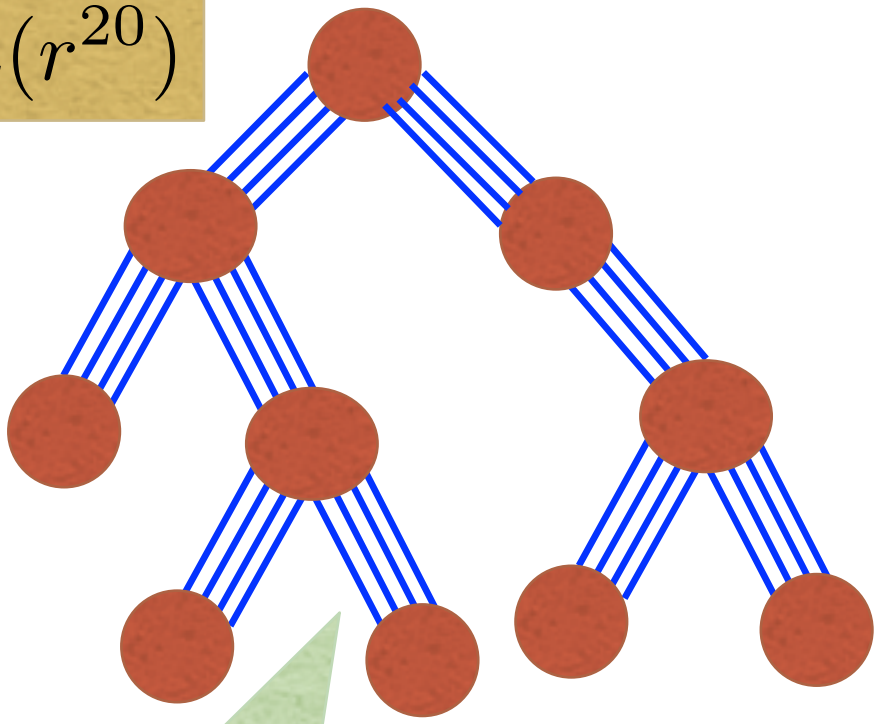
Tree-of-Sets System

$p = \Omega(r^{20})$



Tree-of-Sets System

$p = \Omega(r^{20})$



at least p/r^4 edges

Any graph of treewidth k

↓

Nice graph of treewidth k/polylog(k)

↓

A family of r disjoint good routers

↓

Weak Tree-of-sets-system

↓

Tree-of-sets-system

↓

Path-of-sets system

↓

Large grid minor

Open Problems

- Improve Grid-Minor theorem parameters
- Tight tradeoff for treewidth decomposition theorem
- Applications of related ideas?
- Similar results for *directed* treewidth?

Thank You!