Independent Sets in Elimination Graphs: Submodular Objective

Chandra Chekuri & Kent Quanrud
UIUC & Purdue
Max Weight Independent Set

\[ G = (V, E) \] undirected graph

\[ S \subseteq V \] is an independent/stable set if \( G[S] \) has no edges

**Problem:** given \( G \) and weights \( w(v), v \in V \)

find max (weight) independent set in \( G \)

NP-Hard, and also very hard to approximate!

No \( \Omega\left(\frac{1}{n^{1-o(1)}}\right) \) approx. unless \( P = NP \) [Hastad99, Zuckerman06]
Special Cases and Applications

• Sparse graphs: bounded degree or degeneracy
• Interval graphs, chordal graphs, …
• Geometric intersection graphs
• Inductively k-independent graphs
• Perfectly k-orientable graphs
• …

Theory but also because of applications
Elimination Graphs

• Inductively \( k \)-independent graphs

\[
\subset
\]

• Perfectly \( k \)-orientable graphs

Capture several classes in parameterized and unified way, both sparse and dense

MWIS has \( \frac{1}{k} \) and \( \frac{1}{2k} \) approximation in above classes

[AADK02]  [KT14]
Elimination Graphs

• Inductively $k$-independent graphs
• Perfectly $k$-orientable graphs

MWIS has $\frac{1}{k}$ and $\frac{1}{2k}$ approximation in above classes

This talk: obtain results when objective is submodular
Submodular Set Functions

Real-valued set function $f : 2^V \rightarrow \mathbb{R}$ is **submodular** if

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \quad \forall A, B$$

Equivalently:

$$f(A + v) - f(A) \geq f(B + v) - f(B) \quad A \subset B, v \notin B$$
Submodular Set Functions

Real-valued set function $f : 2^V \rightarrow R$ is submodular if

$$f(A + v) - f(A) \geq f(B + v) - f(B) \quad A \subset B, \forall v \notin B$$

$f$ is monotone if $f(A) \leq f(B)$ for all $A \subseteq B$

$f$ is non-negative if $0 \leq f(A)$ for all $A$

Assume $f(\emptyset) = 0$
Motivation

Many positive results and applications on constrained submodular set function maximization

\[
\max f(S) \text{ where } S \text{ is independent}
\]

- matroid and matroid like constraints
- knapsack constraints and packing integer programs
- ...

Extend positive results to graph independent sets
Max Submod Independent Set

\( G = (V, E) \) undirected graph

\( S \subseteq V \) is an independent/stable set if \( G[S] \) has no edges

**Problem:** given \( G \) and submodular \( f : 2^V \rightarrow R_+ \)

\[ \max f(S) \text{ where } S \text{ is independent set in } G \]
Result 1

- **k-perfectly orientable graphs:** randomized algorithm that yields $\frac{1}{e(k+1)}$ approx. for non-neg submod funcs. Slightly better for monotone functions (see paper)

- Based on multilinear relaxation and rounding approach
- Randomized and inefficient
- But can be parallelized
Result 2

- **Inductively k-independent graphs:** deterministic algorithm that yields \( \frac{1}{k+1+2\sqrt{k}} \) approx. for monotone functions and randomized \( \Omega\left(\frac{1}{k}\right) \) approx. for non-monotone functions.

- Based on primal-dual technique
- Very simple and efficient
- First deterministic approx. for several special cases such as interval graphs, pseudo-disk graphs etc.
Inspiration and Take Aways

Results are based on combining existing ideas but …

• Graph classes are *unifying* and *interesting*

• *Primal-dual* for submod max was first done by [Levin-Wajc21] for streaming application. This paper shows applications for offline
  • For interval graphs first deterministic approx. and ratio of $\frac{1}{4}$ matches previous best randomized [Feldman13]

• Connections to streaming algorithms

• Several open problems (see paper)
Interval Graphs

• Given $n$ intervals $I_1, I_2, \ldots, I_n$ where $I_i = [a_i, b_i]$

• Defines *intersection graph*: vertices correspond to intervals and two intervals connected iff they overlap

Max weight independent set in interval graphs is easy. Several ways to see this but take graph theoretic view

They are chordal and have perfect elimination ordering

Vertices can be ordered as $v_1, v_2, ..., v_n$ such that for each $i$, $N(v_i) \cap \{v_i, v_{i+1}, ..., v_n\}$ is a clique

$N(v_i)$ is set of neighbors of $v_i$
Sort intervals by right end point

B A E D C F G
Inductively k-independent graphs

[AADK02, YB12]

G is inductively k-independent if there is a vertex ordering \( v_1, v_2, ..., v_n \) such that for each \( i \), max independent set of \( G[N(v_i) \cap \{v_i, v_{i+1}, ..., v_n\}] \leq k \)

Easier condition: \( G[N(v_i) \cap \{v_i, v_{i+1}, ..., v_n\}] \) can be covered by \( k \) cliques

Chordal/interval graphs: \( k = 1 \) due to single clique

Matchings in graphs: line-graph has \( k = 2 \) since for any edge its neighbors are covered by two cliques (any ordering will work!)
t-interval graphs

n objects: object is a collection of t disjoint intervals, form intersection graph

Max-Indep-Set hard and non-obvious to approximate even when t = 2 [BNSS06]
**k-perfectly-orientable graphs**

[KT14]

\(G=(V, E)\) has an orientation \(H=(V, A)\) such that for each vertex \(v\), \(N^+_H(v)\) can be covered by \(k\) cliques

\(N^+_H(v)\) is out-neighborhood of \(v\)

Orientation is powerful. Exist 2-perfectly orientable graphs that are not inductively \(\sqrt{n}\) −independent

t-interval graphs are 2t-perfectly-orientable
LP Relaxation

Assume ordering $v_1, v_2, \ldots, v_n$ is given.

Variable $x_i$ whether $v_i$ is in independent set.

Constraints are simple:

\[ x_i + \sum_{j \in A_i} x_j \leq k \text{ for all } i \]
\[ x_i \in [0,1] \text{ for all } i \]

$A_i: N(v_i) \cap \{v_i, v_{i+1}, \ldots, v_n\}$ or $N_H^+(v_i)$
Multilinear Relaxation Based Algorithms

\[
\max F(x) \text{ such that } \\
x_i + \sum_{j \in A_i} x_j \leq k \quad \text{for all } i \\
x_i \in [0,1] \text{ for all } i
\]

\[A_i: N_H^+(v_i)\]

Can obtain \((1-1/e)\) approx. for relaxation via continuous greedy

Round \(x\) via contention resolution scheme
Randomized Rounding

Round $x$ via contention resolution scheme

Very simple and natural here (implicit in [Feldman13] for interval graphs)

- Pick each $v_i$ with probability $\frac{c x_i}{k}$ for some constant $c$. Let $R$ be random set

- Discard $v_i$ from $R$ if $R$ has any vertex from neighborhood $N_H^+(v_i)$

- Output altered solution $R'$

*(balance parameters appropriately to get best ratio)*
Primal-Dual

Inspired by [Levin-Wajc’21]

Only for inductively k-independent

Want LP relaxation so use “concave extension” $f^+(x)$

$max f^+(x)$ such that

$$x_i + \sum_{j \in A_i} x_j \leq k \quad for \ all \ i$$

$$x_i \in [0,1] \ for \ all \ i$$
\( f^+(x) \) hard to evaluate but won’t need to solve since we are using primal-dual

\[
\text{max } f^+(x) \quad \text{such that} \\
\quad x_i + \sum_{j \in A_i} x_j \leq k \quad \text{for all } i \\
\quad x_i \in [0,1] \quad \text{for all } i
\]

Rewrite above as big LP from definition of \( f^+(x) \)
Primal and Dual

\[
\begin{align*}
\max \sum_{S \subseteq V} \alpha_L f(L) \\
\sum_{L \subseteq V} \alpha_L &= 1 \\
\sum_{L \ni v_i} \alpha_L &= x_i \quad i \in [n] \\
x_i + \sum_{v_j \in A_i} x_j &\leq k \quad i \in [n] \\
x_i &\geq 0 \quad i \in [n]
\end{align*}
\]

\[
\begin{align*}
\min \mu + k \sum_{i=1}^{n} y_i \\
\mu + \sum_{v_i \in L} z_i &\geq f(L) \quad L \subseteq V \\
y_i + \sum_{v_j \in B_i} y_j &\geq z_i \quad i \in [n] \\
y_i &\geq 0 \quad i \in [n]
\end{align*}
\]
Primal-Dual Algorithm

Related to local-ratio and stack-based algorithms for MWIS (see paper for refs). Restrict attention to monotone functions. Non-neg requires more work.

Two phases

- **Phase 1**: process vertices in order $v_1, v_2, \ldots, v_n$ and choose a subset $S$ stored in a stack

- **Phase 2**: Process $S$ in reverse order by popping. Obtain a subset $S'$ that is an independent set
Phase 1

- Initialize stack $S$ to empty
- For $i = 1$ to $n$ do
  - Process $v_i$
  - $C_i = S \cap N(v_i)$  // the vertices in $S$ that conflict with $v_i$
  - If $f_S(v_i) \geq (1 + \beta) \sum_{v_j \in C_i} w_j$ then
    - // $v_i$ is significantly better than conflict set so add
    - $S.push(v_i)$
    - $w_i = f_S(v_i)$  // $w_i$ is value when added
Phase 2

- Initialize $S'$ to empty
- While $S$ is not empty
  - $v = S.pop$
  - If $S' + v$ is an independent set then
    - $S' \leftarrow S' + v$
- Output $S'$
Open Problem

\(\frac{1}{4}\) approx. for interval graphs when \(f\) is monotone

- Via primal-dual
- And via multilinear relaxation

Can we do better?
Thank You!
Backup Slides
Interval Graphs

• Given \( n \) intervals \( I_1, I_2, \ldots, I_n \) where \( I_i = [a_i, b_i] \)

• Defines intersection graph: vertices correspond to intervals and two intervals connected iff they overlap