Maximizing A Submodular Set Function (Revisited)

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Motivation

[Nemhauser-Wolsey-Fisher 78]
An analysis of the approximations for maximizing submodular set functions I, Math Programming, 1978

[Fisher-Nemhauser-Wolsey 78]

Interesting applications
[CK’00,FHR03,CK’04,MV’04,CP’05,FGMS’06,...]
Outline

- Problem Definition
- Greedy algorithm’s performance
- Example application: generalized assignment
- LP Relaxation to improve upon Greedy
Submodular set function

\( E : \) discrete ground set
\( f: 2^E \to \mathbb{R}^+ \)

\( f \) is submodular iff

\[
f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad \text{for all } A, B \subseteq V
\]

equivalently

\[
f(A+\{e\}) - f(A) \leq f(B+\{e\}) - f(B) \quad \text{if } B \subseteq A
\]
Monotone Submodular set function

$E$ : discrete ground set

$f$: $\mathcal{P}(E) \rightarrow \mathbb{R}^+$

$f$ is submodular iff

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad \text{for all } A, B \subseteq V$$

monotone (non-decreasing)

$$f(A) \geq f(B) \quad \text{if } B \subseteq A$$

$$f(\emptyset) = 0$$
Max Submodular Set Function

Discrete set \( E \)

Submodular set function \( f: 2^E \rightarrow \mathbb{R}^+ \)

\[
\max_{s \subseteq V} f(S)
\]

s.t

\( S \in \text{allowed sets} \)

\( f \) given by a valuation oracle

allowed sets also specified as an oracle
Max Submodular Set Function

Discrete set $E$

Submodular set function $f: 2^E \rightarrow \mathbb{R}^+$

$$\max_{S \subseteq V} f(S)$$

subject to

$S \in \text{independent sets } (I \subseteq 2^E)$

$f$ given by a valuation oracle

independent sets also specified as an oracle
Greedy Algorithm

start with $S = \emptyset$

repeat

\[ A = \{ e \mid S+e \text{ is independent} \} \]

if $A = \emptyset$ then STOP and output $S$

\[ e' = \arg\max_{e \in A} f(S+e) - f(S) \]

$S = S + e'$
When is Greedy optimal?

[Edmonds 6?]
f is modular (linear): \( f(S) = \sum_{e \in S} w(e) \)

\[
\begin{align*}
\text{max } f(S) \\
\text{s.t} \\
S \text{ is an independent set in a matroid } M \text{ on } E
\end{align*}
\]

Example: max/min weight spanning tree (Kruskal’s algorithm)
Matroid

discrete set $E, \mathcal{I} \subseteq 2^E$

$M = (E, \mathcal{I})$ is a matroid if

- $\emptyset \in \mathcal{I}$
- $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$
- $A, B \in \mathcal{I}$ and $|A| < |B|$ implies, there exists $e \in B \setminus A$ s.t $A + e \in \mathcal{I}$

$A \in \mathcal{I}$ is “independent”

A is a *cycle* if it is minimally non-independent

(there is $e$ s.t $A - e$ is independent)
Examples of matroids

- \( \mathcal{I} = \{ A \mid |A| \leq k \} \)
- \( E_1, E_2, \ldots, E_h \) a partition of \( E \), \( k_1, \ldots, k_h \in \mathbb{Z}^+ \)
  \( \mathcal{I} = \{ A \mid |A \cap E_i| \leq k_i \} \)
- Laminar families (trees) with integer capacities
- Graphic matroid. \( G = (V, E) \). \( A \) is independent iff \( A \) induces a forest
- \( E \) is a collection of vectors in a vector space, \( A \) is independent if the vectors are linearly indep
When is Greedy optimal?

[Edmonds 6?]

\[ f \text{ is linear} \]

\[ \max f(S) \]

\[ s.t \]

\[ S \text{ is an } \textit{independent} \text{ set in a matroid } M \text{ on } E \]
Greedy for $p$-independence families

[Jenkyns 76, Korte-Hausman 78]

If $f$ is linear Greedy is a $1/p$ approximation alg. for $p$-independence families

Example: Greedy for maximum weight matching in a graph is a $\frac{1}{2}$ approximation
p-independence families

Discrete Set $E$, Independence family $\mathcal{F} \subseteq 2^E$

Downward closed: $A \in \mathcal{F}$ and $B \subset A \Rightarrow B \in \mathcal{F}$

$A \in \mathcal{F}$ is “independent”

$A$ is a circuit or minimally non-independent if there is $e$ s.t

$A - \{e\}$ is independent

$p$-independent family: for every $A \in \mathcal{F}$, $A + \{e\}$ has at most $p$ circuits

or

there exists $B = \{e_1, e_2, ..., e_p\}$ s.t $A - B + e \in \mathcal{F}$
Examples: p-independence families

\( p = 2 \)

Intersection of 2 matroids
Example: \( G = (V_1, V_2, E) \) is a *bipartite* graph
\( A \subseteq E \) is independent if \( A \) induces a *matching* in \( G \)

(not matroid intersection)
\( G = (V, E) \) is a graph. \( A \subseteq E \) is independent if \( A \) induces a *matching* in \( G \)
Greedy and implicit set systems

Often $E$ is implicit and $|E|$ is exponential in input problem size.

Greedy can be implemented if the step
$$e' = \arg\max_{e \in A} f(S+e) - f(S)$$
can be implemented in polynomial time.

$\alpha \leq 1$ approx for oracle $\Rightarrow$ Greedy is $\alpha/p$ approx for linear $f$.
On to submodularity

Examples of submodular set functions

- set systems
- from matroids (rank functions etc)
- cut functions in graphs (not monotone)
- ...

Submodularity from Set Systems

\(X_1, X_2, ..., X_n\) subsets of \(\mathcal{U}\)
\(E = \{1, 2, ..., n\}\)

for \(A \subseteq E\), let \(X(A) = \bigcup_{i \in A} X_i\) (elements covered by \(A\))

for \(A \subseteq E, \ e \in \mathcal{U}, \ c(A, e) = \{i \in A \mid e \in X_i\}\)

Coverage: \(f(A) = |X(A)| = \) Multi-coverage: \(f(A) = \sum_{e \in X(A)} \max\{k(e), |c(A, e)|\}\)
Weighted coverage: \(f(A) = \sum_{e \in X(A)} w(e)\)
Weighted max coverage: \(f(A) = \sum_{e \in X(A)} \max_{i \in c(A, e)} w(e, i)\)
Set Systems

\[ E \]

\[ \mathcal{X}_i \]

\[ \mathcal{U} \]
Maximizing Submodular Set Functions

\[
\text{max } f(S) \\
\text{s.t } \\
|S| \leq k
\]

NP-hard (for example via Maximum Coverage)
How good is Greedy?

\[
\begin{align*}
\text{max } & f(S) \\
\text{s.t } & |S| \leq k
\end{align*}
\]

Greedy yields a \( 1-1/e \) approximation

[Nemhauser-Wolsey-Fisher 78]
Special case: Maximum Coverage

$X_1, X_2, \ldots, X_n$ subsets of ground set $U$

Pick $k$ out of $n$ sets to maximize size of their union

$E = \{1, 2, \ldots, n\}$

$f(A \subset E) = | \bigcup_{i \in A} X_i |$

$max f(S)$

$|S| \leq k$
Maximum Coverage

$X_1, X_2, \ldots, X_n$ subsets of ground set $U$
Pick $k$ out of $n$ sets to maximize size of their union

$E = \{1, 2, \ldots, n\}$
$f(A \subset E) = \left| \bigcup_{i \in A} X_i \right|$

Unless $NP \subset DTIME(n^{\log \log n})$, there is no $1-1/e - \varepsilon$ approximation for Maximum Coverage [Feige98]
Greedy for p-independence families

\[
\max f(S)
\]

s.t

\( S \) is independent in \( \mathcal{I} \)

[Fisher-Nemhauser-Wolsey 78]

Greedy is a \( \frac{1}{p+1} \) approximation

\( \frac{1}{2} \) when \((E, \mathcal{I})\) is a matroid

(\( \alpha \) appox for oracle, Greedy is \( \alpha/(p+\alpha) \) approx)
Other results in [NWF78, FNW78]

- Simpler Greedy for interesting special class
- $1/2$ approx for local search algorithm for $p=1$
- bad performance for local search when $p > 1$
- LP formulation/analysis for some special cases
- Examples
Example: Multiple Knapsack and Generalized Assignment

- Multiple knapsack (uniform):
  - \( m \) knapsacks of size \( B \) each.
  - \( n \) items. item \( j \) has size \( s_j \), profit \( p_j \).

- Multiple knapsack (non-uniform):
  - knapsack sizes \( B_1 < B_2 < ... < B_m \)

- Generalized assignment:
  - item \( j \) has size \( s_{ij} \) and profit \( p_{ij} \) to go into knapsack \( i \)
  - knapsack sizes 1 (wlog)
Greedy for Multiple Knapsack

Use single knapsack FPTAS iteratively
How good is it?
Greedy for Multiple Knapsack

Use single knapsack FPTAS iteratively

How good is it?

[C-Khanna ’00]

Greedy is $1-1/e - \varepsilon$ approx for uniform

$(\max f(S) \text{ s.t } |S| \leq m)$

Greedy is $1/2 - \varepsilon$ approx for non-uniform

$(\max f(S) \text{ s.t } S \text{ is independent in a matroid})$
Generalized Assignment (GAP)

$m$ knapsacks
Each item $j$ has per knapsack size and profit
$s_{ij}$: size of item $j$ in knapsack $i$
$p_{ij}$: profit of packing item $j$ in knapsack $i$
Pack items into bins to maximize profit

$X$ - set of items
GAP as Max f(S)

\[ E_i = \{ A \subseteq X \mid s_i(A) \leq 1 \} \]

A \in E_i \text{ implies items in } A \text{ fit into knapsack } i

\[ E = \bigcup_{i=1}^{m} E_i \]

Given \( A \subseteq E \),
\[ f(A) = \sum_{j \in X} \max_{i: S \in E \cap E_i, j \in S} p_{ij} \]

GAP: \( \max f(S) \text{ s.t } |S \cap E_i| \leq 1 \) (simple partition matroid constraint)
Set system for GAP
Greedy for Generalized Assignment

\( p_j = \) current accrued profit of item \( j \)
\( p_j = 0 \) for all \( i \) (all items have initial profit 0)

For \( i = 1 \) to \( m \) do

\[
p'_{ij} = \max \{0, p_{ij} - p_j\} \quad \text{(residual profit of item } j\text{)}
\]

Solve knapsack for bin \( i \) using \( p'_{ij} \) s as profits (all items)

Let \( S_i \) be items assigned in knapsack

If \( j \) is in \( S_i \), \( p_j = p_j + p'_{ij} \)

Remark: an item can be packed multiple times!
Greedy for Gen Assignment

Greedy is a $\frac{1}{2} - \varepsilon$ approximation!
Local search also $\frac{1}{2} - \varepsilon$ approximation!

LP based $\frac{1}{2}$ approx [C-Khanna’00] (using [Shmoys-Tardos’91])

[Fleischer-Goemans-Mirrokni-Sviridenko’06]
Improved $1-\frac{1}{e}-\varepsilon$ using stronger LP

[Feige-Vondrak’06]
$1-\frac{1}{e}+\delta$ using same LP
Other applications

Greedy + additional ideas

k-repairman
Orienteering (with Time Windows)
Group Steiner

...

[C-Kumar 04, C-Pal 05]
Is Greedy the best one can do?

Greedy gives $1/(p+1)$ approximation

For large $p$, unless $P = NP$, no algorithm better than $\Omega(\log p/p)$ [Hazan-Safra-Schwartz ’03] (max $p$-dimensional matching, $f$ is modular)
(Single) Matroid constraint

Given $E$, $f$ and $M=(E, \mathcal{I})$

$$\max_{S \subseteq V} f(S)$$

s.t

$S \in \mathcal{I}$

Question: can we do better than Greedy $(1/2)$? $(1-1/e)$ is best possible even for simple $|S| \leq k$
(Single) Matroid constraint

Given $E$, $f$ and $M = (E, \mathcal{I})$

$$\max_{S \subseteq V} f(S)$$

s.t

$S \in \mathcal{I}$

**Theorem**: $1 - 1/e$ for some restricted $f$ such as $f$ arising from set systems.
(Single) Matroid constraint

**Theorem:** $1 - 1/e$ for some restricted $f$ such as $f$ arising from set systems.

With some work, $1 - 1/e$ for generalized assignment (known from [Fleisher-Goemans-Mirrokni-Sviridenko’06] essentially a reinterpretation)
(Single) Matroid constraint

Theorem: can obtain $1 - \frac{1}{e}$ for some restricted $f$ such as $f$ arising from set systems.

Proof idea: use LP relaxation and rounding (inspired by pipage-rounding [Ageev-Sviridenko 04])
(Single) Matroid constraint

What does $f$ need to satisfy?

$f(y)$ defined for $y \in \{0,1\}^n$

Extend $f$ to $[0,1]^n$ (relaxation to fractional values)

For each $y \in [0,1]^n$ if $E_y[f(y)] \geq \alpha \ f(y)$, then $\alpha$ approximation (easy for set system type problems)
Useful settings

Maximum coverage under following constraints

- $|S| \leq k$ (Greedy gives $1-1/e$)
- Sets are colored and can pick at most $k_i$ colors from sets of color $i$ (partition matroid)
- Color class constraints + total $\# \leq k$ (laminar family constraint)
- Sets are associated with edges of a graph and picked sets induce a tree/forest
LP Relaxation

\[
\begin{align*}
\max & \quad f(S) \\
S & \in \mathcal{I}
\end{align*}
\]

\(y_i \in \{0,1\}\) to indicate if \(i \in E\) is in solution

\[
\begin{align*}
\max & \quad f(y) \\
y & \in P(M) \\
y_i & \in \{0,1\}
\end{align*}
\]

\(P(M)\) : polytope of matroid \(M\)
LP Relaxation

\[ \text{max } f(y) \]
\[ y(A) \leq r(A) \quad S \subseteq E \]
\[ y_i \in [0,1] \quad i \in E \]

Need to extend \( f \) from \( \{0,1\}^n \) to \( [0,1]^n \)

Easy/natural for coverage type problems
LP Relaxation

Need to extend $f$ from $\{0,1\}^n$ to $[0,1]^n$

$$f(y) = \sum_{e \in \mathcal{U}} \max \{1, \sum_{e \in x_i} y_i\}$$

What about a general $f$ given as oracle?

[FNW’78] formulation if $f$ is matroid type

Observation: integer valued $f$ is of matroid type using [Helgason’74] (weakly polynomial)

(useful in other contexts?)
LP Rounding

If $y_i \in \{0, 1\}$, reduce problem to smaller instance
Can assume $y$ is completely fractional

set $A$ is tight if $y(A) = r(A)$
A, B tight $\Rightarrow$ $A \cup B$, $A \cap B$ also tight
$\Rightarrow$ minimal tight sets are disjoint
LP Rounding

set $A$ is tight if $y(A) = r(A)$

$A, B$ tight $\Rightarrow A \cup B, A \cap B$ also tight

$\Rightarrow$ minimal tight sets are disjoint
LP Rounding

set $A$ is tight if $y(A) = r(A)$

$A, B$ tight $\Rightarrow A \cup B, A \cap B$ also tight

$\Rightarrow$ minimal tight sets are disjoint

$y_i = y_i + \epsilon, \; y_j = y_j - \epsilon$
set $A$ is tight if $y(A) = r(A)$

$A, B$ tight $\Rightarrow A \cup B, A \cap B$ also tight

$\Rightarrow$ minimal tight sets are disjoint

Claim: New minimal tight set \textit{strictly contained} in $A$
LP Rounding

Choose $\epsilon > 0$ or $< 0$?

According to potential function $E[f(y)]$, which ever is greater

Lemma: $E[f(y)]$ is convex in $\epsilon$ for any two variables $y_i, y_j$ if $f$ is submodular
Conclusions

Remember $\frac{1}{p+1}$ approx ratio of Greedy submodularity + independence constraints quite general

LP relaxation yields $1 - \frac{1}{e}$ for one matroid constraint for some $f$ – better than $\frac{1}{2}$ given by Greedy (need to solve LP, not always possible for exponential sized systems)

Open Problem: can we obtain $1 - \frac{1}{e}$ for all submodular $f$? (exists an LP relaxation for integer valued $f$ using [Helgason 74] )
Thank You!
Simple question

Given graph $G = (V, E)$
$y(e) \in [0,1]$ for $e \in E$
$y$ belongs to the spanning tree polytope
(that is $y$ is a convex combination of spanning trees)

Randomly choose each $e$ with prob $y(e)$
What is the expected number of connected components in resulting graph?
Conjecture: $\leq \frac{n}{e}$ (known ?)
Integer valued f - polymatroids

Polymatroids have nice polyhedral description
[Edmonds]
(useful for LP formulations)

[Helgason74]
Every polymatroid $f: 2^V \rightarrow \mathbb{Z}^+$ has an underlying matroid $M = (V', \mathcal{I})$ s.t $f$ is the rank function of $M$

$|V'| = \sum_{i \in V} f(i)$ (weakly polynomial)
(Single) Matroid constraint

What does $f$ need to satisfy?

$f(y)$ defined for $y \in \{0,1\}^n$

Extend $f$ to $[0,1]^n$ (relaxation to fractional values)

For each $y \in [0,1]^n$ if $E_y[f(y)] \geq \alpha \cdot f(y)$, then $\alpha$ approximation
Rounding in Matroids

Set $A$ is tight if $y(A) = r(A)$

Pick minimal tight set $H$ and let $y_i, y_j$ be fractional in $H$

For some small $\varepsilon > 0$,

- $y_i = y_i + \varepsilon, y_j = y_j - \varepsilon$
- $y_i = y_i - \varepsilon, y_j = y_j + \varepsilon$

are both feasible

Choose max $\varepsilon$ and do one of above

Claim: new minimal tight set will be contained in $H$

Process will converge – eventually some $y_i = 1$ or 0
Rounding in Matroids

Claim: new minimal tight set will be contained in $H$

$A, B$ tight implies $A \cap B$ and $A \cup B$ are tight

Minimal tight sets do not intersect
Choose which option?

[Ageev-Sviridenko] pipage rounding

Define potential function $F(y)$ with following properties

- $F(y) = f(y)$ for integer $y$
- $F(y) \geq \alpha f(y)$ for all $y$
- $g_{ij}(\varepsilon, y)$ is convex in $\varepsilon$ when changing of two variables $y_i$ and $y_j$
Choose which option?

[Ageev-Sviridenko] pipage rounding

Define potential function $F(y)$ with following properties

- $F(y) = f(y)$ for integer $y$
- $F(y) \geq \alpha f(y)$ for all $y$
- $g_{ij}(\varepsilon, y)$ is convex in $\varepsilon$ when changing of two variables $y_i$ and $y_j$
Rounding contd ... 

$y_1$ is obtained from $y$ by $y_i + \varepsilon$, $y_j - \varepsilon$

$y_2$ is obtained from $y$ by $y_i - \varepsilon$, $y_j + \varepsilon$

By convexity condition,

$F(y_1) \geq F(y)$ or $F(y_2) \geq F(y)$

Hence after rounding

$f(y') = F(y') \geq F(y) \geq \alpha f(y) \geq \alpha \text{OPT}$
Potential Function?

How do we get a potential function?

[Ageev-Sviridenko] simple explicit potential function for maximum coverage

\[ \sum_e (1 - \prod_{e \in X_i} y_i) \]

Natural potential function: \( F(y) = E_y[f(y)] \)

independently set \( i \) to 1 with probability \( y_i \)
Potential Function?

Natural potential function: \( F(y) = E_y[f(y)] \)

Lemma: \( g_{ij}(\varepsilon, y) \) is convex if \( f \) is submodular!

Only thing to do: prove \( F(y) \geq \alpha \ f(y) \)

For many set system type problems \( \alpha = 1-1/e \)  
(easy to prove)