Performance Guarantees for the TSP with a Parameterized Triangle Inequality

Michael A. Bender* Chandra Chekuri†

Abstract

We consider the approximability of the TSP problem in graphs that satisfy a relaxed form of triangle inequality. More precisely, we assume that for some parameter $\tau \geq 1$, the distances satisfy the inequality $\text{dist}(x, y) \leq \tau \cdot (\text{dist}(x, z) + \text{dist}(z, y))$ for every triple of vertices x, y, and z. We obtain a 4τ-approximation and also show that for some $\epsilon_0 > 0$ it is NP-hard to obtain a $(1 + \epsilon_0\tau)$-approximation for all $\tau \geq 1$. Our upper bound improves upon the earlier known ratio of $(\tau^2 + \tau)$ [1] for all values of $\tau > 3$.

Keywords: approximation algorithm, traveling salesman problem, relaxed triangle inequality.

Introduction. The approximability of the Traveling Salesman Problem (TSP) depends on the particular version of the problem. Specifically, if the distances are Euclidean, the TSP admits a polynomial-time approximation scheme [3, 11]. If the distances form a metric, then the problem is MAX-SNP-hard [13] and 3/2 is the best approximation ratio known [4]. For arbitrary (symmetric) distances, however, the problem cannot be approximated to within any factor. The negative result follows by a reduction to the problem of finding a Hamiltonian cycle in a graph.

It seems natural that slight variations in the distances of a problem should affect the solution to the problem only slightly, and this should hold even if a perturbed instance of metric TSP no longer obeys the Δ-inequality. This intuition suggests parameterizing TSP instances by how *strongly* they disobey the Δ-inequality. In particular, for parameter $\tau \geq 1$,

$$\text{dist}(x, z) \leq \tau \cdot (\text{dist}(x, y) + \text{dist}(y, z)).$$

If $\tau = 1$, the distances satisfy the triangle inequality. For $\tau > 1$, the distances satisfy the *relaxed triangle inequality.*

*Department of Computer Science, State University of New York at Stony Brook, Stony Brook, NY 11794-4400, USA. Email: bender@cs.sunysb.edu. Supported in part by ISX Corporation and Hughes Research Laboratories.

†Bell Labs, 600 Mountain Avenue, Murray Hill, NJ 07974, USA. Email: chekuri@research.bell-labs.com.

Andreae and Bandelt [2] show a $(3\tau^2/2 + \tau/2)$-approximation for the parameterized triangle inequality TSP (Δ_τ-TSP). By refining the approach in [2], Andreae [1] obtains an improved ratio of $(\tau^2 + \tau)$. In this paper, we provide a simple algorithm that obtains a 4τ-approximation to the Δ_τ-TSP. This result is optimal to within a constant factor.

The Problem. An instance of the problem is a weighted graph G consisting of n vertices (representing cities). Distances between vertices satisfy the Δ_τ-inequality and the length of an optimal TSP tour is denoted OPT. The objective of the problem is to find (in polynomial time) a tour having length at most $f(\tau)\cdot$OPT, where $f(\tau)$ is a slowly-growing function of τ.

Intuition. Unfortunately, an approximation algorithm on a graph obeying the Δ-inequality does not automatically translate into an approximation algorithm on a graph obeying the parameterized Δ-inequality. To understand why, notice an advantage of the triangle inequality: shortcutting vertices never increases distances, no matter how many vertices are avoided. Thus, the TSP on a *subset* of vertices is a lower bound on the TSP of all the vertices.

This advantage is lost with the parameterized Δ-inequality. Shortcutting over ℓ vertices may increase the distance substantially: the shortcut distance may depend on the parameter τ^ℓ. Thus, the TSP on a subset of vertices may be dramatically larger than the TSP on all the vertices.

We illustrate this difficulty by examining two well known approximation algorithms for the Δ-TSP. A 2-approximation is obtained as follows:

1. Find a minimum spanning tree (MST) for G.
2. Create a multigraph by duplicating each edge in the MST.
3. Find an Eulerian tour.
4. Shortcut over nodes to obtain a Hamiltonian cycle.

The Christofides heuristic [4] yields a 3/2-approximation:

1. Find a MST for G.
2. Identify the nodes of odd degree and find a minimum-cost perfect matching for these nodes.
3. Find an Eulerian tour in the multigraph consisting of the MST and the perfect matching.
4. Shortcut the Eulerian tour to obtain a Hamiltonian cycle.

The performance guarantees follow because the MST is a lower bound on OPT and the length of minimum-cost perfect matching is a lower bound on OPT/2.

Δ_τ-TSP. Now consider what happens to these algorithms for the Δ_τ-TSP. The MST remains a lower bound on OPT, however the minimum cost perfect matching on the odd-degree vertices is no longer a lower bound on OPT/2.

The Δ_τ-TSP algorithm of [2] finds an MST and doubles edges to create an Eulerian tour. Then, it obtains a Hamiltonian cycle by shortcutting. In the algorithm, at most 2 consecutive nodes are removed from any part of the tour, and thus the approximation ratio is $O(\tau^2)$. It is never necessary to shortcut *more* than 2 nodes, because the cube of any connected graph is Hamiltonian [16]. (The cube of G, denoted G^3, contains edge (u, v) if there is a path from u to v in G of at most 3 edges.) This approach cannot obtain an approximation better than $O(\tau^2)$, because there are instances when the MST is a factor of $O(\tau^2)$ smaller than OPT.

$O(\tau)$-Approximation Algorithm. In order to obtain a better approximation ratio, we begin with a tighter lower bound. A graph S is 2-*node-connected* if the deletion of any node from S leaves it connected. Any Hamiltonian cycle is 2-node-connected. Thus, the length of a *minimum-cost* 2-*node-connected subgraph* is a lower bound on OPT. Notice that it is NP-hard to find the length of the minimum-cost 2-node-connected subgraph. However, a 2-approximation can be computed in polynomial time [8, 15].

Let S^2 denote the square of graph S ((u, v) is an edge in S^2 if there is a path in S between u and v using at most 2 edges). We first provide an 8τ-approximation and then show how it can be improved to obtain a 4τ-approximation. The 8τ-approximation algorithm is described below.

1. Find an approximation to the minimum-cost 2-node-connected subgraph of G. Call this graph S.
2. Find a Hamiltonian cycle in S^2.

In 1970, Fleischner proved that the square of a 2-node-connected graph is Hamiltonian [6, 7], and Lau, in his Ph.D. thesis, provided a constructive proof [9, 10] that finds a Hamiltonian cycle in $O(n^2)$ time. We note that Fleischner's result is applied to the TSP in a similar way by Parker and Rardin [14], who show a 2-approximation for the bottleneck-TSP.

Lemma 1 *Any Hamiltonian cycle in S^2 yields an 8τ-approximation.*

Proof. The graph S has weight at most $2 \cdot$ OPT. Notice that it is not sufficient for the analysis to bound the total weight of the edges in S^2, because the weight could be a factor of $O(n\tau)$ larger than the weight of the edges in S. Thus, we need a more careful analysis.

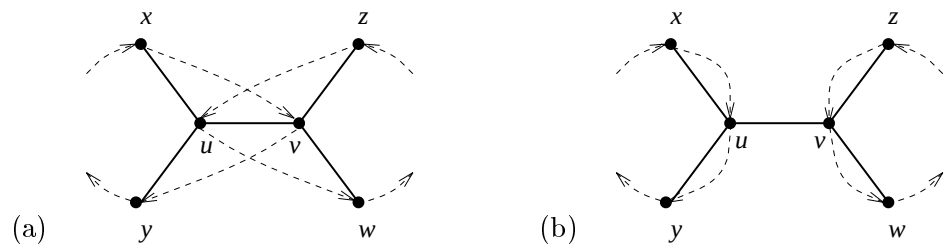

Figure 1: In (a), edge $\{u, v\}$ is crossed 4 times by cycle C. In (b), cycle C is modified so that $\{u, v\}$ is not crossed.

Let C be a Hamiltonian cycle in S^2, and consider an orientation of this cycle. Let (u, w) be a (directed) edge in C. We define a relation *cross* between the directed edges of C and the undirected edge of S. An edge $(u, w) \in C$ crosses its undirected counterpart $\{u, w\}$ provided $\{u, w\} \in S$. If $\{u, w\} \notin S$ then (u, w) crosses all edges of S on paths of length two from u to w. Cycle C is said to cross an undirected edge $\{u, v\} \in S$ k times where k is the number of distinct edges of C that cross $\{u, v\}$. Figure 1 shows two different cycles and how they cross the edge $\{u, v\}$.

Notice that no Hamiltonian cycle crosses any edge of S more than 4 times. This is because each node u in S is incident to exactly two edges in C, one edge entering u and one edge leaving u (see Figure 1(a)). This observation immediately yields an 8τ-approximation. □

We obtain an improved 4τ-approximation by finding a Hamiltonian cycle that does not cross any edge of S more than twice. Our modified algorithm is given below.

TSP-APPROX (G)

1. Find an approximation to the minimum-cost 2-node-connected subgraph of G. Call this graph S.
2. Find a Hamiltonian cycle C in S^2.
3. Transform C into a cycle C' that crosses each edge of S at most twice.

Lemma 2 *A Hamiltonian cycle in S^2 that crosses each edge of S at most twice can be found in polynomial time.*

Proof. Let C be any Hamiltonian cycle in S^2. Let C_e be the number of times C crosses an edge $e \in S$. If $C_e \leq 2$ for all $e \in S$ we are done. Otherwise let $e = \{u, v\} \in S$ be such that $C_e > 2$.

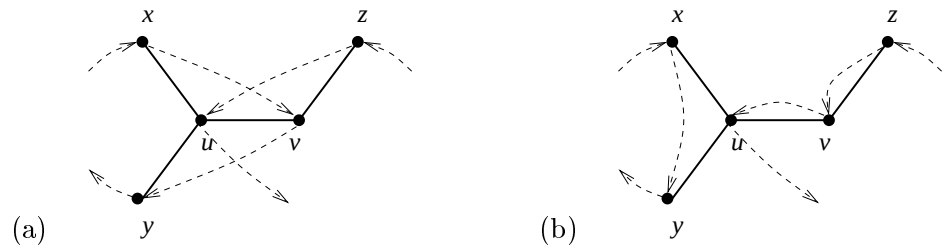

Figure 2: In (a), edge $\{u, v\}$ is crossed 3 times by cycle C. In (b), cycle C is modified so that $\{u, v\}$ is crossed once.

We obtain a new cycle C' such that $C'_e \leq 2$, and for all edges $f \neq e$, $C'_f \leq C_f$. Iterating this procedure yields the desired Hamiltonian cycle. As we mentioned before $C_e \leq 4$ for any cycle, therefore we consider two cases based on whether $C_e = 4$ or $C_e = 3$. It is easy to see that for any edge $\{u, w\} \in S$, $C_{\{u,v\}} \leq 2$ if either (u, v) or (v, u) is an edge of C.

Case 1 is depicted in Figure 1(a), where $\{u, v\}$ is crossed 4 times. The solid lines in the figure represent edges in S. The dotted lines represent edges in C. Cycle C can be modified as shown in Figure 1(b), so that $\{u, v\}$ is not crossed.

Case 2 is shown in Figure 2(a), where $\{u, v\}$ is crossed 3 times. Notice that the edge out of u is in S for otherwise we would be in Case 1. Cycle C can be modified as in Figure 2(b) so that $\{u, v\}$ is crossed only once.

In both cases it is easy to see that $C'_f \leq C_f$ for every edge. It is also clear that identifying the two cases and transforming as described above can be accomplished in polynomial time. □

Theorem 1 *Algorithm* TSP-APPROX *is a* 4τ*-approximation algorithm.*

Running time: We now analyze the running time of our algorithm. The bottleneck is computing the 2-approximation for the minimum-cost 2-node-connected subgraph, which requires $O(n^5)$ time [15]. A Hamiltonian cycle in S can be found in $O(n^2)$ time [9]. The transformations outlined in Lemma 2 take $O(1)$ time per edge. Thus, the overall running time is $O(n^5)$. Notice that the running time of our algorithm is independent of τ and consequently the algorithm runs well on any graph regardless of how strongly it disobeys the triangle inequality. Only the performance guarantee depends on τ, and the value of τ can be determined in time $O(n^3)$ by examining all triples of nodes in G.

Theorem 2 *There exists some fixed constant* $\epsilon > 0$ *such that it is NP-hard to approximate* Δ_τ-TSP *to within a factor better than* $(1 + \epsilon\tau)$.

Proof. We use a reduction from TSP with distances one and two which is MAX-SNP-hard [13]. The proof in [13] establishes that there exists an infinite class of graphs $\mathcal{G}$ with the following properties. Let $G = (V, E)$ be a graph in $\mathcal{G}$. There exists a fixed constant $\epsilon > 0$ such that either G is Hamiltonian, or every Hamiltonian cycle in the complete graph on V requires at least ϵn edges from $\bar{E} = V \times V - E$. Further, given a graph from $\mathcal{G}$ it is NP-complete to distinguish which of the above two cases applies. MAX-SNP-hardness of one-two TSP follows from this. We use the same class of graphs to obtain our result. Given a graph $G = (V, E)$ from $\mathcal{G}$ we create an instance of Δ_τ-TSP in the obvious way by making the graph complete with edges in E having cost 1, and the rest having cost 2τ. If G is Hamiltonian the cost of an optimal TSP tour in G is n, otherwise it is at least $(1 - \epsilon)n + 2\tau\epsilon n$. Since it is NP-complete to distinguish between these two cases the theorem follows. □

Open Problems. Our upper and lower bounds are within a constant factor of each other; however, the gap is large. Can we obtain a $k\tau$ approximation for some k smaller than 4? One approach might be to combine the two steps in our algorithm in some clever way. Any improvement in the approximation ratio for the minimum-cost 2-node-connected subgraph problem would also improve the ratio for our problem.

It is natural to consider the same relaxation of triangle inequality for the Asymmetric TSP problem. Although an $O(\log n)$ approximation is known with the triangle inequality, no approximation factor of the form $O(\tau^\ell \log n)$ is known for the relaxed version for any constant ℓ.

The relaxed triangle inequality is useful in practice (see [5, 12] for applications to pattern matching of images) and finding robust algorithms whose performance degrades only mildly with τ is of interest.

Acknowledgments.
We warmly thank Santosh Vempala for helpful discussions. We also thank Joe Mitchell and the anonymous referees for their comments.

References

[1] T. Andreae. On the traveling salesman problem restricted to inputs satisfying a relaxed triangle inequality. Technical Report 74, Mathematisches Seminar, University of Hamburg, 1998.

[2] T. Andreae and H.-S. Bandelt. Performance guarantees for approximation algorithms depending on parametrized triangle inequalities. *SIAM Journal of Discrete Mathematics*, 8(1):1–16, February 1995.

[3] S. Arora. Polynomial-time approximation schemes for Euclidean TSP and other geometric problems. In *37th IEEE Symposium on Foundations of Computer Science*, pages 2–12, 1996. To appear in *Journal of the ACM*.

[4] N. Christofides. Worst-case analysis of a new heuristic for the Traveling Salesman Problem. Technical Report 338, Graduate School of Industrial Administration, Carnegie Mellon University, 1976.

[5] R. Fagin and L. Stockmeyer. Relaxing the triangle inequality in pattern matching. *International Journal of Computer Vision*, 30(3):219–31, 1998.

[6] H. Fleischner. On spanning subgraphs of a connected bridgeless graph and their application to DT graphs. *Journal of Combinatorial Theory*, 16:17–28, 1974.

[7] H. Fleischner. The square of every two-connected graph is Hamiltonian. *Journal of Combinatorial Theory*, 16:29–34, 1974.

[8] S. Khuller and B. Raghavachari. Improved approximation algorithms for uniform connectivity problems. *Journal of Algorithms*, 21:434–450, 1996.

[9] H.T. Lau. *Finding a Hamiltonian Cycle in the Square of a Block*. PhD thesis, McGill University, February 1980.

[10] H.T. Lau. Finding EPS-graphs. *Monatshefte für Mathematik*, 92:37–40, 1981.

[11] J.S.B. Mitchell. Guillotine subdivisions approximate polygonal subdivisions: a simple polynomial-time approximation scheme for geometric TSP, k-MST, and related problems. *SIAM Journal of Computing*, 28(4):1298–309, 1999.

[12] W. Niblack, R. Barber, W. Equitz, M. Flickner, E. Glasman, D. Petkovic, and P. Yanker. The QBIC project: querying images by content using color, texture, and shape. In *Proceedings of SPIE Conference on Storage and Retrieval for Image and Video Databases*, volume 1908, pages 173–181, 1993.

[13] C.H. Papadimitriou and M. Yannakakis. The traveling salesman problem with distances one and two. *Mathematics of Operations Research*, 18(1):1–16, February 1993.

[14] R. Gary Parker and Ronald L. Rardin. Guaranteed performance heuristics for the bottleneck traveling salesman problem. *Operations Research Letters*, 2(6):269–272, 1984.

[15] M. Penn and H. Shasha-Krupnik. Improved approximation algorithms for weighted 2 & 3 vertex connectivity augmentation problems. *Journal of Algorithms*, 22:187–196, 1997.

[16] M. Sekanina. On an ordering of the set of vertices of a connected graph. *Publication of the Faculty of Sciences of the University of Brno*, 412:137–142, 1960.