Flow-Cut Gaps for Integer and Fractional Multiflows

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September 6, 2012

Abstract

Consider a *routing problem* instance consisting of a *demand graph* $H = (V, E(H))$ and a *supply graph* $G = (V, E(G))$. If the pair obeys the cut condition, then the *flow-cut gap* for this instance is the minimum value C such that there exists a feasible multiflow for H if each edge of G is given capacity C. It is well-known that the flow-cut gap may be greater than 1 even in the case where G is the (series-parallel) graph $K_{2,3}$. In this paper we are primarily interested in the "integer" flow-cut gap. What is the minimum value C such that there exists a feasible integer valued multiflow for H if each edge of G is given capacity C? We formulate a conjecture that states that the integer flow-cut gap is quantitatively related to the fractional flow-cut gap. In particular this strengthens the well-known conjecture that the flow-cut gap in planar and minor-free graphs is $O(1)$ [15] to suggest that the integer flow-cut gap is $O(1)$. We give several technical tools and results on non-trivial special classes of graphs to give evidence for the conjecture and further explore the "primal" method for understanding flow-cut gaps; this is in contrast to and orthogonal to the highly successful metric embeddings approach. Our results include the following:

- Let G be obtained by series-parallel operations starting from an edge st, and consider orienting all edges in G in the direction from s to t. A demand is *compliant* if its endpoints are joined by a directed path in the resulting oriented graph. We show that if the cut condition holds for a compliant instance and $G + H$ is Eulerian, then an integral routing of H exists. This result includes as a special case, routing on a ring but is not a special case of the Okamura-Seymour theorem.
- Using the above result, we show that the integer flow-cut gap in series-parallel graphs is 5. We also give an explicit class of routing instances that shows via elementary calculations that the flow-cut gap in series-parallel graphs is at least $2 - o(1)$; this is motivated by and simplifies the proof by Lee and Raghavendra [21].
- The integer flow-cut gap in k-Outerplanar graphs is $c^{O(k)}$ for some fixed constant c.
- A simple proof that the flow-cut gap is $O(\log k^*)$ where k^* is the size of a node-cover in H; this was previously shown by Günlük via a more intricate proof [14].

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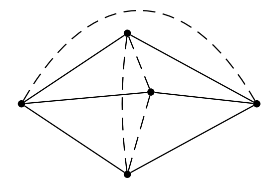
1 Introduction

Given a (undirected) graph $G = (V, E)$ a *routing* or *multiflow* consists of an assignment $f : \mathcal{P} \to R_+$ where $\mathcal{P}$ is the set of simple paths in G and such that for each edge e, $\sum_{P \in P(e)} f_P \leq 1$, where $P(e)$ denotes the set of paths containing e. Given a *demand graph* $H = (V, E(H))$ such a routing *satisfies* H if $\sum_{P \in P(u,v)} f_P = 1$ for each $g = uv \in E(H)$, where $P(u, v)$ denotes paths with endpoints u and v (one may assume a simple demand graph without loss of generality). If such a flow exists, we call the instance *routable*, or say H is routable in G. Edges of G and H are called *supply edges* and *demand edges* respectively. The above notions extend naturally if each supply edge e is equipped with a capacity u_e and each demand edge g is equipped with a demand d_g. If u is an integral vector, we denote by G_u, the graph obtained by making u_e copies of each edge e. H_d is defined similarly. We call the routing f *integral* (resp. *half-integral*) if each f_P (resp. $2f_P$) is an integer.



For any set $S \subseteq V$ we denote by $\delta_G(S)$ the set of edges with exactly one end in S, and the other in $V - S$. We define $\delta_H(S)$ similarly. (For graph theory notation we primarily follow Bondy and Murty [4].) The supply graph G satisfies the *cut condition* for the demand graph H if $|\delta_G(S)| \geq |\delta_H(S)|$ for each $S \subset V$. We sometimes say that the pair G, H satisfies the cut-condition. Clearly the cut condition is a necessary condition for the routability of H in G. The cut-condition is not sufficient as shown by the well-known example where $G = K_{2,3}$ is a series-parallel graph with a demand graph (in dotted edges) as shown in the figure.

Given a graph G and a real number $\alpha > 0$ we use αG to refer to the graph obtained from G by multiplying the capacity of each edge of G by α. Given an instance G, H that satisfies the cut-condition, the *flow-cut gap* is defined as the smallest $\alpha \geq 1$ such that H is routable in αG; we also refer to α as the *congestion*. We denote this quantity by $\alpha(G, H)$. Traditional combinatorial optimization literature has focused on characterizing conditions under which the cut-condition is sufficient for (fractional, integral or half-integral) routing, in other words the setting in which $\alpha(G, H) = 1$; see [30] for a comprehensive survey of known results. Typically, these characterizations involve both the supply and demand graphs. A prototypical result is the Okamura-Seymour Theorem [25] that states that the cut-condition is sufficient for a half-integral routing if G is a planar graph and all edges of H are between the nodes of a single face of G in some planar embedding. The proofs of such result rely on what we will term "primal-methods" in that they try to directly exhibit routings of the demands, rather than appealing to dual solutions.

On the other hand, since the seminal work of Leighton and Rao [20] on flow-cut gaps for uniform and product multiflow instances, there has been an intense focus in the algorithms and theoretical computer science community on understanding flow-cut gap results for classes of graphs. This was originally motivated by the problem of finding (approximate) sparse cuts. A fundamental and important connection was established in [23, 3] between flow-cut gaps and metric-embeddings. More specifically, for a graph G, let $\alpha(G)$ be the largest flow-cut gap over all possible capacities on the edges of G and all possible demand graphs H. Also let $c_1(G)$ denote the maximum, over all possible edge lengths on G, of the minimum *distortion* required to embed the finite metric on the nodes of G (induced by the edge lengths) into an ℓ_1-space. Then the results in [23, 3] showed that $\alpha(G) \leq c_1(G)$ and subsequently [15] showed that $\alpha(G) = c_1(G)$. Using Bourgain's result that $c_1(G) = O(\log |V|)$ for all G, [23, 3] showed that $\alpha(G) = O(\log |V(G)|)$, and further refined it to prove that $\alpha(G, H) = O(\log |E_H|)$. Numerous subsequent results have explored this connection to obtain a variety of flow-cut gap results. The proofs via metric-embeddings are "dual"-methods since they work by embedding the metric induced by the dual of the linear program for the maximum concurrent multicommodity flow. The embedding approach has been successful in obtaining flow-cut gap results (amongst several other algorithmic applications) as well as forging deep connections between various areas of discrete and continuous mathematics. However, this approach does not directly give us integral routings even in situations when they do exist.

In this paper we are interested in the *integer* flow-cut gap in undirected graphs. Given G, H that satisfy the cut-condition, what is the smallest α such that H can be integrally routed in αG? Is there a relationship between the (fractional) flow-cut gap and the integer flow-cut gap? A result of Nagamochi and Ibaraki relates the two gaps in *directed* graphs. Let $G = (V, A)$ and $H = (V, R)$ be a supply and demand digraph, respectively. We call (G, H) *cut-sufficient* if for each capacity function $u : A \to \mathcal{Z}^+$ and demand function $d : R \to \mathcal{Z}^+$, G_u, H_d satisfying the cut-condition implies the existence of a fractional multiflow for H_d in G_u.

Theorem 1.1 ([24]) *If (G, H) is cut-sufficient, then for any integer capacity vector u and integer demand vector d such that G_u, H_d satisfy the cut condition, there is an integer multiflow for H_d in G_u.*

The above theorem does not extend to the undirected case. Consider taking G to be a cycle and H to be a complete graph. Then it is known that (G, H) is cut-sufficient but we are not guaranteed an integral flow for integer valued u and d. An example is when G is a 4 cycle with unit capacities and H consists of two crossing edges with unit demands. However when G is a cycle, there is always a half-integral routing of H_d in G_u whenever (G_u, H_d) satisfies the cut-condition and u and d are integer valued. We may therefore ask if a weaker form of Theorem 1.1 holds in undirected graphs. Namely, where we only ask for half-integral flow instead of integral flows.

One case where one does get such a half-integral routing in undirected graphs is the following. Consider the case when $G = H$; if the pair (G, G) is cut-sufficient we simply say that G is cut-sufficient. It turns out that this is precisely the class of K_5-minor free graphs (Seymour [31]; cf. Corollary 75.4d [30]). Moreover we have the following.

Theorem 1.2 (Seymour) *If G is cut-sufficient, then for any nonnegative integer weightings u, d on $E(G)$ for which G_u, G_d satisfies the cut condition, there is a half-integral routing of G_d in G_u. Moreover, if $G_u + G_d$ is Eulerian, then there is an integral routing of G_d.*

In this paper we ask more broadly, whether the fractional and integral flow-cut gaps are related even in settings where the flow-cut gap is greater than 1. We formulate the conjecture below. Let $G = (V, E)$ and $H = (V, R)$ be a supply and demand graph. We call (G, H) a *β-congestion pair* if they satisfy the following. For each capacity function $u : E \to \mathcal{Z}^+$ and demand requirement $d : R \to \mathcal{Z}^+$, if the pair G_u, H_d satisfies the cut-condition, then there is a fractional multiflow for H_d in G_d with congestion β.

Conjecture 1.3 (Gap-Conjecture) *Does there exist a global constant C such that the following holds? Let (G, H) be a β-congestion pair. If a capacity function $u : E \to \mathcal{Z}^+$ and demand requirement $d : R \to \mathcal{Z}^+$ satisfy the cut condition, then there is an* integer *multiflow for H_d in G_d with congestion $C\beta$.*

We do not currently know if the statement holds for $\beta = 1$ congestion pairs and with $C = 2$. This would generalize Seymour's theorem mentioned above, which establishes this for $(\beta = 1)$-congestion pairs of the form (G, G). There are several natural weakenings of the conjecture that are already unknown. For instance, one may allow C to depend on a class of instances (such as planar or series parallel supply graphs). A weaker conjecture (and more plausible) is to bound the *integer flow cut gap* as some function $g(\beta)$, e.g., $g(\beta) = O(\text{poly}(\beta))$. Previously, other conjectures relating fractional and integer multiflows were shown to be false. For instance, Seymour conjectured that if there is a fractional multiflow for G, H, then it implies a half-integer multiflow. These conjectures have been strongly disproved (see [30]). Note that our conjecture differs from the previous ones in that we relate the flow-cut gap values for hereditary classes of instances on G, H.

The Gap-Conjecture has several important implications. First, it would give structural insights into flows and cuts in graphs. Second, it would allow fractional flow-cut gap results obtained via the embedding-based approaches to be translated into integer flow-cut gap results. Finally, it would also shed light on the approximability of the congestion minimization problem in special classes of graphs. In congestion minimization we are given G, H and are interested in the least α such that αG has an integer routing for H. Clearly, the congestion required for a fractional routing is a lower bound on α; moreover this lower bound can be computed in polynomial time via linear programming. Almost all the known approximation guarantees are with respect to this lower bound; even in directed graphs an $O(\log n/\log\log n)$ approximation is known via randomized rounding [27]. In general undirected graphs, this problem is hard to approximate to within an $\Omega(\log\log n)$-factor [1]. However, for planar graphs and graphs that exclude a fixed minor, it is speculated that the problem may admit an $O(1)$ approximation. The Gap-Conjecture relates this to the conjecture of Gupta et al. [15] that states that the fractional flow-cut gap is $O(1)$ for all graphs that exclude a fixed minor. Thus the congestion minimization problem has an $O(1)$ approximation in minor-free graphs if the Gupta et al. conjecture and the Gap-Conjecture are both true. We also note that an $O(1)$ gap between fractional and integer multiflows in planar graphs (or other families of graphs) would shed light on the Gap-Conjecture.

Our current techniques seem inadequate to resolve the Gap-Conjecture. It is therefore natural to prove the Gap-Conjecture in those settings where we do have interesting and non-trivial upper bounds on the (fractional) flow-cut gap. Note that the conjecture follows easily when G and H are unrestricted (complete graphs). In this case the flow-cut gap is $\Omega(\log n)$; one may consider G, a bounded degree expander, with H, a uniform multiflow [20]. As for the upper bound, randomized rounding shows that the integer flow-cut gap is $O(\log n)$. Now, if G is a complete graph and H is a complete graph on a subset of k nodes of G, then the flow-cut gap for such instances is $\Omega(\log k)$; this easily follows from the expander example above. In this setting, the upper bound for the fractional flow-cut gap improves to $O(\log k)$, as shown in [23, 3]. One can also obtain an improvement for the integer flow-cut gap but one cannot employ simple randomized rounding. In [11] it is shown that the integer flow-cut gap for these instances is $O(\text{polylog}(k))$ (thus satisfying the weaker gap conjecture); this relies on the results in [9, 17].

In a sense, the Gap-Conjecture is perhaps more relevant and interesting in those cases where the flow-cut gap is $O(1)$. We focus on series-parallel graphs and k-Outerplanar graphs for which we know flow-cut gaps of 2 [6] and c^k (for some universal constant c) [8] respectively. Proving flow-cut gaps for even these restricted families of graphs has taken substantial technical effort. In this paper we affirm that one can prove similar bounds for these graphs for the integer flow-cut gap. For instance, in series parallel instances, we show that the integer flow-cut gap is at most 5 (and we conjecture it is 2).

Overview of results and techniques: In this paper we focus especially on applying primal methods to two classes of graphs for which the flow-cut gap is known to be $O(1)$: series parallel graphs and k-Outerplanar graphs.

The first proof that series parallel instances had a constant flow-cut was given in [15]; subsequently a gap of 2 was shown in [6]. This latter upper bound is tight since it is shown in [21] that there are instances where the gap is arbitrarily close to 2. We give a simpler proof of the lower bound in this paper that is based on an explicit (recursive) instance and elementary calculations — our proof is inspired by [21] but avoids their advanced metric-embeddings machinery.

In Section 4.1 we show that for series-parallel graphs the integer flow-cut gap is at most 5. The primal-method has generally been successful in identifying restrictions on demand graphs for which the cut-condition implies routability. We follow that approach and identify several classes of demands in series-parallel graphs for which cut-condition implies routability (see Sections 3.1 - 3.4). The main class exploited to obtain the congestion 5 result, are the so-called compliant demands (Section 3.4). However, the critical base case for compliant demands boils down to determining classes of demands on K_{2m} instances for which the cut condition implies routability. In fact, for K_{2m} instances, we are able to give a complete characterization of demand graphs H for which (K_{2m}, H) is cut sufficient — see Section 3.3. This forbidden minor characterization was subsequently extended to all series-parallel graphs [7] (it does not completely characterize cut-sufficiency in general however).

One ingredient we use is a general proof technique for "pushing" demands similar to what has been used in previous primal proofs; for instance in the proof of the Okamura-Seymour theorem [25]. We try to replace a demand edge uv by a pair of edges ux, xv to make the instance simpler (we call this *pushing to* x). Failing to push, identifies some tight cuts and sometimes these tight cuts can be used to shrink to obtain an instance for which we know a routing exists. This contradiction means that we could have pushed in the first place.

In [8], an upper bound of c^k (for some constant c) is given for the flow-cut gap in k-Outerplanar graphs.

In this paper (Section 4.3), we show that the integer flow cut gap in this case is $c^{O(k)}$. In this effort, we explicitly employ a second proof ingredient which is a simple *rerouting* lemma that was stated and used in [10] (see Section 4.2). Informally speaking the lemma says the following. Suppose H is a demand graph and for simplicity assume it consists of pairs $s_1t_1, \ldots, s_kt_k$. Suppose we are able to route the demand graph H' consisting of the edges $s_1s_1', t_1t_1', \ldots, s_ks_k', t_kt_k'$ in G where $s_1', t_1', \ldots, s_k', t_k'$ are some arbitrary intermediate nodes. Let H'' be the demand graph consisting of $s_1't_1', \ldots, s_k't_k'$. The lemma states that if G, H satisfies the cut-condition and the aforementioned routing exists in G then $2G, H''$ satisfies the cut-condition. Clearly we can compose the routings for H' and H'' to route H. The advantage of the lemma is that it allows us to reduce the routing problem on H to that in H'' by choosing H' appropriately. This simple lemma and its variants give a way to prove approximate flow-cut gaps effectively.

The rerouting lemma sometimes leads to very simple and insightful proofs for certain results that may be difficult to prove via other means — see [10]. In this paper we give two applications of the lemma. We give (in Section 4.4) a very short and simple proof of a result of Günlük [14]; he refined the result of [23, 3] and showed that $\alpha(G, H) = O(\log k^*)$ where k^* is the node-cover size of H. Clearly $k^* \leq |E_H|$ and can be much smaller. We also show that the integer flow-cut gap for k-Outerplanar graphs is $c^{O(k)}$ for some universal constant c; in fact we show a slightly stronger result (see Section 4.3). Previously it was known that the (fractional) flow-cut gap for k-Outerplanar graphs is c^k [8].

Our integer flow-cut gap results imply corresponding new approximation algorithms for the congestion minimization problem on the graph classes considered. Apart from this immediate benefit, we feel that it is important to complement the embedding-based approaches to simultaneously develop and understand corresponding tools and techniques from the primal point of view. As an example, Khandekar, Rao and Vazirani [17], and subsequently [26], gave a primal-proof of the Leighton-Rao result on product multicommodity flows [20]. This new proof had applications to fast algorithms for finding sparse cuts [17, 26] as well as approximation algorithms for the maximum edge-disjoint path problem [29].

2 Basics and Notation

We first discuss some basic and standard reduction operations in primal proofs for flow-cut gaps and also set up the necessary notation for series-parallel graphs.

2.1 Some Basic Operations Preserving the Cut Condition

We present several operations that turn an instance G, H satisfying the cut condition into smaller instances with the same property. We call an instance G, H *Eulerian* if $G + H$ is Eulerian; we also seek to preserve this property.

For $S \subseteq V$, the capacity of the cut $\delta_G(S)$, is just $|\delta_G(S)|$ (or sum of capacities if edges have capacities). Similarly, the demand of such a cut is $|\delta_H(S)|$. Hence the surplus is $\sigma(S) = |\delta_G(S)| - |\delta_H(S)|$. The set S, and cut $\delta(S)$, is called *tight* if $\sigma(S) = 0$. The *cut condition* is then satisfied for an instance G, H if $\sigma(S) \geq 0$ for all sets S. One may naturally obtain "smaller" routing instances from G, H by performing a contraction of a subgraph of G (not necessarily a connected subgraph) and removing loops from the resulting G', and in the resulting demand graph H'. It is easily checked that if G, H has the cut condition, then so does any contracted instance.

We call a subset $A \subseteq V(G)$ *central* if both $G[A]$ and $G[V - A]$ are connected. We refer to a central set also as a central cut. The following is well-known cf. [30].

Lemma 2.1 *G, H satisfy the cut condition if and only if the surplus of every central set is nonnegative.*

1-cut reduction: This operation takes an instance where G has a cut node v and consists of splitting G into nontrivial pieces determined by the components of $G - v$. Demand edges f with endpoints x, y in distinct components are replaced by two demands xv, yv and given over to the obvious instance. One easily checks that each resulting instance again satisfies the cut condition. A simple argument also shows that the Eulerian property is maintained in each instance if the original instance was Eulerian.

Parallel reduction: This takes as input an instance with a demand edge f and supply edge e, with the same endpoints. The reduced instance is obtained by simply removing f, e from H and G respectively. If the supply and demand graphs are unweighted, trivially the new instance satisfies the cut condition and is Eulerian if G, H was. (If supply graph has capacities and demand graph has demands then e can be removed from G or f from H by appropriately reducing the demand of f or the capacity of e.)

Slack reduction: This works on an instance where some edge e (in G or H) does not lie in any tight cut. In this case, if $e \in G$, we may remove e from G and add it to H. If $e \in H$, we may add two more copies of e to H. Again, this trivially maintains the cut condition and the Eulerian property.

Push operations: Such an operation is usually applied to a demand edge xy whose endpoints lie in distinct components of $G - \{u, v\}$ for some 2-cut u, v. *Pushing a demand xy to u* involves replacing the demand edge xy by the two new demands xu, uy. Such an operation clearly maintains the Eulerian property but it may not maintain the cut condition. We have

Fact 2.2 *Pushing a demand xy to u maintains the cut condition in an Eulerian instance if and only if there is no tight cut $\delta(S)$ that contains u but none of x, y, v.*

We call the preceding four operations *basic*, and we generally assume throughout that our instance is reduced in that we cannot apply any of these operations. In particular, we may generally assume that G is 2-node connected.

2.2 Series Parallel Instances

A graph is *series-parallel* if it can be obtained from a single edge graph st by repeated application of two operations: series and parallel operations. A *parallel* operation on an edge e in graph $G = (V, E)$ consists of replacing e by $k \geq 1$ new edges with the same endpoints as e. A *series* operation on an edge consists of replacing e by a path of length $k > 1$ between the same endpoints. Series-parallel graphs can also be characterized as graphs that do not contain K_4 as minor.

A *capacitated graph* refers to a graph where each edge also has an associated positive integer capacity. For purposes of routing, any such edge may be viewed as a collection of parallel edges. Conversely, we may also choose to identify a collection of parallel edges as a single *capacitated edge*. In either case, for a pair of nodes u, v, we refer to the *capacity* between them as the sum of the capacities of edges with u, v as endpoints. For a pair of nodes u, v a *bridge* is either a (possibly capacitated) edge between u, v or it is a subgraph obtained from a connected component of $G - \{u, v\}$ by adding back in u, v with all edges between u, v and the component. In the latter case, the bridge is *nontrivial*. A *strict cut* is a pair of nodes u, v with at least 2 nontrivial bridges and at least 3 bridges.

Lemma 2.3 *If G is a 2-node-connected series-parallel graph, then either it is a capacitated ring, or it has a strict 2-cut.*

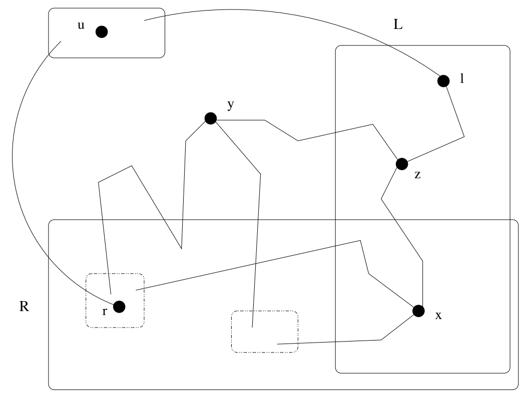
Proof: Suppose that G has no strict 2-cut. Let $e_1, e_2, \ldots, e_k$ be the result of the first parallel operation on the original edge st. Let $P_i = a_1, a_2, \ldots, a_l$ be the path obtained after subdividing e_i if there is ever a nontrivial series operation applied to e_i (where the a_j's are the edges). Since G is 2-node-connected, $k \geq 2$. If $k = 2$, then each edge of P_1, P_2 results in a capacitated edge in G and hence G is a capacitated ring. If this were not the case, then some a_j has a parallel operation followed by a series operation, and hence the ends of a_j would form a strict cut in G. So suppose that $k \geq 3$. Clearly at most one e_i is subdivided, say P_1, or else s, t is a strict cut. Again, either each edge of P_1 becomes a capacitated edge. Otherwise any series operation to some a_p results in its endpoints inducing a strict 2-cut. ∎

The following lemma is useful in applying the push operation (cf. Fact 2.2).

Lemma 2.4 *Let u, v be a pair of nodes in a series parallel graph, and let l, r be a 2-cut separating u from v. Let L be a central set containing l, but not u, r and v; and let R be a central set containing r, but not u, l and v. Then $L \setminus R$ and $R \setminus L$ are central.*

Proof: First, we prove that $V \setminus (R \setminus L) = (V \setminus R) \cup L$ is connected. This is the case since $(V \setminus R)$ and L are each connected, and both contain l.

Then, we prove that $R \setminus L$ is connected. Let R_1 be the connected component of $R \setminus L$ that contains r. Assume $R \setminus L$ contains another connected component R_2. The 2-cut l, r separates u from v, so $V - l, r$ contains at least two connected components C_u and C_v, containing u and v respectively. Since R_2 contains neither l nor r, it is entirely in a connected component C of $V - l, r$. Assume without loss of generality that C is not C_u. Since R is connected, there is a path P in R from R_1 to R_2; choose R_2 so that P is minimal. It follows that $P = (r_1, Q, r_2)$ where $r_i \in R_i$, and Q is a subpath of $R \cap L$. Note that $P \cup R_2$ is disjoint from C_u.



Since R_1 and R_2 are outside of L, there is a path P' outside of L connecting them. This path may have the form $a_1, Q_1, I_2, Q_2, \ldots, Q_p, a_2$ where $a_i \in R_i$, each Q_i is a path in $V \setminus (R \cup L)$ and each I_j is a path in some component (other than R_1, R_2 in the graph induced by $R \setminus L$. Once again P' is disjoint from C_u, since it contains neither l, r and hence is included in C. Since the internal nodes of P' lie entirely in $V \setminus (R \cup L)$ and P's lie in $R \cap L$, the paths are internally node-disjoint paths connecting R_1, R_2 in the graph $G - C_u$.

Since $G[L]$ is connected, there is a path P_1 joining l to an internal node of P within L; since $G[V \setminus R]$ is connected, there is a path P_2 in this graph connecting l to some internal node of P'. Choose these paths minimal, and let x be the first point of intersection of P_1 with P, and y be the first intersection of P_2 with P'. Once again, note that $P_1 \cup P_2 \setminus l$ does not contain l, r and hence is contained in $G - C_u$ since $y, x \in C$. Choose $z \in P_1$ such that the subpath of P_2 from z to y does not intersect P_1. Clearly $z \in L \setminus R$ by construction. We now have constructed a $K_4 - e$ minor (in fact a homeomorph) on the nodes z, x, y and $r \in R_1$. This graph is also contained within C, except for r and possibly z if $z = l$. We may now extend this to a K_4 minor using C_u and the subpath of P_1 joining l, z.

So $R \setminus L$ is connected, so it is central. $L \setminus R$ is central by the same argument. ■

3 Instances where the Cut Condition is Sufficient for Routing

3.1 Fully Compliant Instances

Let G be a series-parallel supply graph and H a demand graph defined on the same set of nodes. An edge e of H is *fully compliant* if $G + e$ is also series-parallel. An instance G, H is fully compliant if $G + e$ is series-parallel for each $e \in E(H)$. We note that H itself may not be series-parallel in fully compliant instances. For instance, we could take G to be a ring and H to be the complete demand graph.

In this section we prove that fully compliant instances G, H are integrally routable if they satisfy the cut condition and $G + H$ is Eulerian. This forms one base case in showing that compliant instances (introduced in Section 3.4) are routable, which in turn will yield our congestion 5 routing result for general series-parallel instances.

We start with several technical lemmas.

Lemma 3.1 *Let G be a 2-node-connected series-parallel graph. A demand edge uv is fully compliant if and only if there is an edge uv in G, or u, v is a 2-cut in G.*

Proof: Suppose that there is no edge uv in G, and that u, v is not a 2-cut in G. Since G is 2-connected, it contains at least 2 node-disjoint paths from u to v. Since there is no edge uv, each of these paths contains at least one node apart from u and v. And since u, v is not a 2-cut, there is a path connecting these two paths in $G - \{u, v\}$. Therefore, uv is not fully compliant, because $G + uv$ contains a K_4.

Suppose $G + uv$ contains a K_4. Then there is no edge uv in G, since it is series-parallel. Let S_1, S_2, S_3 and S_4 be the sets of nodes in $G + uv$ that form the K_4 minor. First, assume that u and v are in different sets. Without loss of generality u and v are in S_1 and S_2 respectively. Since S_3 and S_4 are adjacent, they are contained in the same connected component of $G - \{u, v\}$, say C. We claim that there cannot be another connected component C' in $G - \{u, v\}$. For if it did, then since G is 2-node-connected, both u and v have an edge to C'. This implies that there is a path between S_1 and S_2 in G that avoids C (and hence S_3 and S_4); then S_1, S_2, S_3, S_4 would form a K_4 minor in G which is impossible since G is series-parallel. Therefore, there is no other connected component in $G - \{u, v\}$, and u, v is not a 2-cut. Now assume that u, v are in the same set, say S_1. Since S_2, S_3, S_4 are adjacent to each other, they are in the same connected component C of $G - \{u, v\}$. We again claim that there cannot be another connected component C' in $G - \{u, v\}$. For if it did, then since G is 2-node-connected, both u and v have an edge to C' and hence there is path between u and v in G that avoids C. This would imply that S_1, S_2, S_3, S_4 would form a K_4 minor in G which is not possible. ■

Lemma 3.2 *Let G be 2-node-connected series-parallel graph. If an edge uv is not fully compliant, then there is a 2-cut separating u from v.*

Proof: Suppose an edge uv is not fully compliant, and there is no 2-cut separating u from v. Then there are three node disjoint paths connecting u and v. Since uv is not fully compliant, there is no edge uv and each of these paths contain at least one node apart from u and v. By Lemma 3.1, u, v is not a 2-cut, so there are paths connecting these paths in $G - \{u, v\}$. This creates a K_4 minor in G, which is impossible. ■

A 2-Cut Reduction: A *partition* of G is any pair of graphs (G_1, G_2) such that: (i) $V(G_1) \cap V(G_2) = \{u, v\}$, for distinct nodes u, v (ii) $E(G)$ is the disjoint union of $E(G_1), E(G_2)$ and (iii) $|V(G_i)| \geq 3$ for each i. Thus any 2-cut admits possibly several partitions, and we refer to any such as a *partition for* $\{u, v\}$. We say that a demand graph H has no demands *crossing* a partition for u, v, if H can be written as a disjoint union $H_1 \cup H_2$ where for $i = 1, 2$, H_i is a subgraph of $H[V(G_i)]$, the demand graph induced by one side of the partition. Note that even if G, H satisfy the cut condition, it may not be the case that G_i, H_i does. It is easily seen however that we may always add some number k_i of parallel edges between u, v in each G_i so that G_i, H_i does have the cut condition. For $i = 1, 2$ the smallest such number is called the *deficit* of the reduced instance G_i, H_i. One easily checks that the deficit of at least one of the reduced instances is 0 if G, H satisfies the cut condition.

Lemma 3.3 *Let G, H satisfy the cut condition and let (G_1, G_2) be a partition for 2-cut $\{u, v\}$ such that H has no demands crossing the partition. Let k_i be the deficit of G_i, H_i; without loss of generality $k_1 \geq 0 = k_2$. Let H_2' be obtained by adding k_1 demand edges between u, v in H_2. We also let $H_1' = H_1$. Let G_1' be obtained by adding k_1 supply edges between u, v to G_1; we also let $G_2' = G_2$. Then G_i', H_i' satisfies the cut condition for $i = 1, 2$. Moreover, if $G + H$ was Eulerian, then so is $G_i' + H_i'$ for $i = 1, 2$. Finally, if there is an integral routing for each instance G_i', H_i', then there is such a routing for G, H.*

Proof: First, let $S \subset V(G_1)$ which defines G_1's deficit. That is the number of demand edges in $\delta_{H_1}(S)$ is k_1 greater than $|\delta_{G_1}(S)|$. Without loss of generality, $u \in S$. Also, we must have $v \notin S$, for otherwise $S \cup V(G_2)$ violates the cut condition for G, H. Clearly, G_1', H_1 now satisfies the cut condition. Next suppose that G_2, H_2' does not obey the cut condition. Then there exists some S' containing u and not v, such that

$$|\delta_{H_2}(S')| + k_1 > |\delta_{G_2}(S')|.$$

But then $S \cup S'$ violates the cut condition for G, H.

Let $G + H$ be Eulerian, and note that all nodes except possibly u, v have even degree in $G'_i + H'_i$ (and in $G_i + H_i$). It is thus sufficient to show that u, v also have even degree. Let p be the parity of u in $G_1 + H_1$. Since any graph has an even number of odd-degree nodes, v must also have parity p in $G_1 + H_1$. Let $s = |\delta_{G_1}(S)|, d = |\delta_{H_1}(S)|$ and so $k_1 = d - s$. Since S separates u, v we have that $d + s = |\delta_{G_1+H_1}(S)|$ has parity p and hence $k_1 = d - s = d + s - 2s$ does as well. That is, the deficit k_1 has parity p and so u and v have even degree in $G'_1 + H'_1$. This immediately implies the same for $G'_2 + H'_2$.

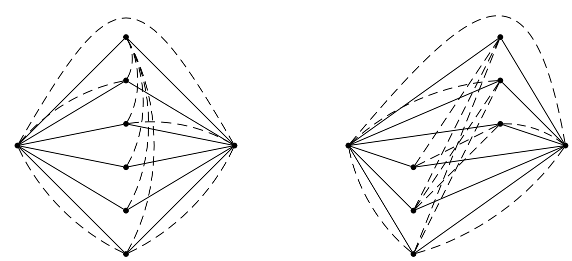
The last part of the lemma is immediate, since each demand from H is routed in $G_1 \cup G_2$ together with the k_1 new supply edges in G_1. However, any such new edge was routed in G_2 together with the demands in H_2, so we can simulate the edge by routing through G_2. ■

Fully Compliant instances are routable: Let G, H satisfy the cut condition where G is series parallel, and each edge in H is compliant (with G). We now show that if $G + H$ is Eulerian, then there is an integral routing of H in G. The proof is algorithmic and proceeds by repeatedly applying the reduction described above and those from Section 2.1. In particular, at any point if there is a slack, parallel or 1-cut reduction we apply the appropriate operation. Thus we may assume that G is 2-node connected, and that each demand or supply edge lies in a tight cut, and no demand edge is parallel to a supply edge.

If G is a capacitated ring, then the result follows from the Okamura-Seymour theorem. Otherwise by Lemma 2.3, there is some strict cut u, v. Thus G has 3 node-disjoint paths P_1, P_2, P_3 between u, v and so there is a partition G_1, G_2 for u, v such that either G_1 or G_2 has 2 node-disjoint paths between u, v. Without loss of generality, for $i = 1, 2$ P_i is contained in G_i. But then there could not be any demand edge crossing the partition. Since if $f \in E(H)$ has one end in $G_1 - \{u, v\}$ and the other in $G_2 - \{u, v\}$, then we would have a K_4 minor in $G + f$. Thus we may decompose using Lemma 3.3 to produce two smaller instances and inductively find routings for them. These two routings yield a routing for G, H by the last part of Lemma 3.3.

3.2 Routable K_{2m} Instances

A K_{2m}-*instance* consists of a supply graph $G = K_{2m}$ with a 2-cut s, t and m nodes $v_1, \ldots v_m$ of degree two, each adjacent to s, t. We may possibly also have an edge between s, t. We also have a demand graph $H = (V = V(G), F)$ on the same node set V, and edge capacities u on G's edges. A *path-bipartite instance* is one where the demands with both ends in the m degree 2 nodes form a bipartite graph. One special case is a so-called *tri-source* instance, where if $v_i v_j \in F$, then either i or j is 1. The figure shows a tri-source and a path-bipartite instance.



Lemma 3.4 *If G, H is a path-bipartite instance satisfying the cut condition and $G + H$ is Eulerian, then there is an integral routing of H in G.*

Proof: If any demand edge f is parallel to an edge $e \in G$, then we may apply the parallel reduction and obtain a smaller Eulerian instance that satisfies the cut condition. Hence we may assume that either there are no st demands, or no st supply edge for otherwise we could reduce the instance. Moreover, we may also assume that there is no demand edge of the form sv_i or $v_i t$. Either we can apply the parallel reduction to eliminate such a demand edge or v_i is only connected to s or t. In the latter case, the instance is not 2-node-connected and we can apply a push operation to a demand edge incident to v_i and reduce; eventually these operations will eliminate v_i. Thus we may assume that every demand edge either joins s, t or is of the form $v_i v_j$ for some $i \neq j$.

Suppose first that some node v_i does not define a tight cut. Consider the new instance obtained by adding a new supply edge between s, t and remove one unit of capacity from each edge incident to v_i. The only central cuts whose supply is reduced is the cut induced by v_i; as this cut was not tight, it still satisfies the Eulerian and cut conditions. The new instance is also smaller in our measure and so we assume that each $\delta(v_i)$ is tight.

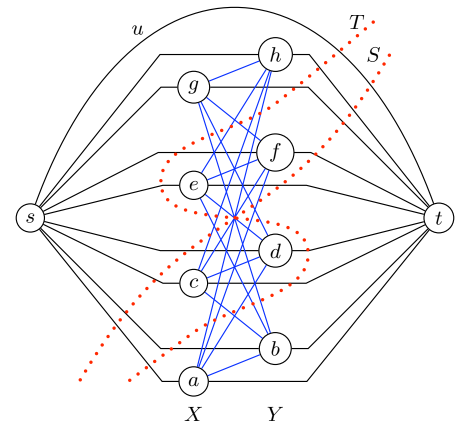
Let X, Y be a bipartition of the degree two nodes for the demands amongst them. Let $r_j = u_{v_j t}$ and $l_j = u_{sv_j}$ for each j. For a subset S of the degree two nodes, we also let $r(S) = \sum_{v_i \in S} r_i$ (similarly for $l(S)$). Hence actual supply out of $\delta_G(X)$ is just $r(X)+l(X)$. We also let $d(i)$ denote the total demand out of v_i; hence we have that $d(i) = r_i + l_i$ for each i. In particular, any subset of X or of Y is tight. Thus if $r(Y) < r(X)$, then $X \cup t$ is a violated cut and so $r(Y) \geq r(X)$. Similarly, $r(X) \geq r(Y)$. Thus $r(X) = r(Y)$ and the same reasoning shows that $l(X) = l(Y)$. Moreover, the above argument shows that there are no st demands or else $X \cup t$ is a violated cut.

Assume that there are two tight cuts S and T separating s from t. We group the nodes of degree two and contract them in eight nodes, by inclusion in X or Y, S or $V \setminus S$, T or $V \setminus T$. Since nodes in X (or Y) are not adjacent to each other, each node is still tight. The result is shown in the adjacent figure.

Some nodes and demand edges may not actually exist.

We denote by l_x (resp. r_x) the capacity of the supply edge between a node x and s (resp. t). We denote by d_{xy} the demand between nodes x and y. We denote by u the capacity of the st supply edge.

The cut induced by S and T are tight which implies:



$$l_a+l_b+l_c+l_d+r_e+r_f+r_g+r_h-d_{af}-d_{ah}-d_{be}-d_{bg}-d_{cf}-d_{ch}-d_{de}-d_{dg}+u = 0,$$

$$l_a + l_b + r_c + r_d + l_e + l_f + r_g + r_h - d_{ad} - d_{ah} - d_{bc} - d_{bg} - d_{cf} - d_{de} - d_{eh} - d_{fg} + u = 0.$$

The surplus of cuts induced by $\{s, d, f, g, h\}$ and $\{s, c, e, g, h\}$ must be positive:

$$l_a + l_b + l_c + r_d + l_e + r_f + r_g + r_h - d_{ad} - d_{af} - d_{ah} - d_{cd} - d_{cf} - d_{ch} - d_{de} - d_{ef} - d_{eh} - d_{bg} + u \geq 0,$$

$$l_a + l_b + r_c + l_d + r_e + l_f + r_g + r_h - d_{ah} - d_{bc} - d_{cd} - d_{cf} - d_{be} - d_{de} - d_{ef} - d_{bg} - d_{dg} - d_{fg} + u \geq 0.$$

If we add these two inequalities and subtract them by the two previous equalities, we get:

$$-2d_{cd} - 2d_{ef} \geq 0,$$

and so $d_{cd} + d_{ef} = 0$. Therefore, for any two distinct st cuts, there is never a demand edge which is on the left of one and on the right of the other.

So any demand edge can be pushed to s or t. By induction, any path-bipartite instance is routable. ■

3.3 K_{2m} instances are cut-sufficient if and only if they have no odd K_{2p} minor

While the preceding result on "bipartite" K_{2m} instances will be sufficient to deduce our congestion 5 result later, we delve a little more deeply into the structure of general K_{2m} instances. This is because they suggest a very appealing conjecture on cut-sufficient series parallel instances (given below); in fact, they formed the basis of a proof of the conjecture in a subsequent article [7].

Recall that a K_{2m}*-instance* is one whose supply graph is of the form $G = K_{2m}$ with a 2-cut s, t and m nodes $v_1, \ldots v_m$ of degree two, each adjacent to s, t. We say an instance has an *odd* K_{2p} *minor* if it is possible to delete or contract supply edges of the instance, and delete demand edges, until we get an instance that is an *odd* K_{2p}, that is, a K_{2p} supply graph, with p odd, such that the two nodes of degree p are connected by a demand edge, and such that the p nodes of degree 2 are connected by an odd cycle of demand edges. One easily sees that an odd K_{2p} instance is not cut-sufficient (refer to the introduction for the definition of cut-sufficiency). Indeed, setting all demands and capacities to 1 defines an instance that satisfies both Eulerian and cut condition, but is not routable. In general, if we only obtain this structure as a minor, we can think of assigning 0 demand (respectively capacity) to the deleted demand (resp. supply) edges, and assign infinite capacity to any contracted supply edge. Hence for any pair (G, H), if it is cut-sufficient, then it cannot contain an odd K_{2p} minor. For general graphs this is not sufficient (cf. examples in [30]). However we conjecture the following.

Conjecture 3.5 *Let G be series parallel. Then a pair (G, H) is cut-sufficient if and only if it has no odd K_{2p} minor.*[1]

Together with the results of the previous section, we now prove that the conjecture holds for K_{2m} instances. In other words, if such an instance has no odd K_{2p} minor, then it is routable. We prove a slightly stronger statement. Let (G, H) be a K_{2m} instance that satisfies the cut condition, is Eulerian and has no odd K_{2p} minor, then there is an integral routing for H in G.

The graphs G and H are capacitated. The proof is by induction on total demand. In the following, we may assume that no demand edge is parallel to a supply edge. Furthermore, we assume as in the proof of Lemma 3.4, that any cut induced by a middle node v_i is tight. If the demand graph does not contain an odd cycle, then it is bipartite and routable by Lemma 3.4. Let us therefore assume that the demand graph contains at least one odd cycle. As the graph should not have any odd K_{2p} minor, there is no demand edge from s to t. We will subsequently prove that at least one demand edge $f = v_i v_j$ can be pushed to s or t, by assuming that none can, and deducing a contradiction. Here, by pushing the edge $f = v_i v_j$ to s we mean that one unit of demand on f can be removed and new demand edges sv_i and sv_j with unit demand are created (pushing to t is similar). Pushing allows us to complete the proof by induction as follows. Once we push a demand edge f to s or t, the resulting instance could possibly have an odd K_{2p} minor. However, the demand edges thus created are parallel to some supply edges, and have only one unit of demand. If we reduce at once the resulting new demand edges with the parallel supply edges, we get a new K_{2m} instance with supply and demand edges that all existed in the original one, and so does not have an odd K_{2p} minor either. As the new instance has a smaller total demand, we can apply induction.

Now we prove the existence of a demand edge that can be pushed. We call a demand edge a *leaf demand edge* if it is the unique demand edge incident to some v_i node. The following lemma is useful in showing that such demand edges can be pushed.

Lemma 3.6 *Let (G, H) be a K_{2m}-instance that satisfies the cut condition and such that any cut induced by a v_i is tight. Then any tight cut separating s from t separates every node v_i from at least one of its adjacent nodes in the demand graph.*

Proof: Suppose there is some tight cut S which separates a degree 2 node v from none of its adjacent nodes in the demand graph. We have $|\delta_G(S)| = |\delta_H(S)|$. Let us consider the cut S' obtained by flipping the side of v in the cut S. Then $|\delta_H(S')| = |\delta_H(S)| + |\delta_H(v)|$, but $|\delta_G(S')| < |\delta_G(S)| + |\delta_G(v)|$, because S separates s from t. Since by assumption $|\delta_H(v)| = |\delta_G(v)|$, the cut condition is not satisfied for S', and so we have a contradiction. ■

As a corollary of the previous lemma, any leaf demand edge in a K_{2m} instance can be pushed to s (or to t), since no tight cut can separate both its end points from s (or from t).

Recall that the demand graph has an odd cycle. We consider the structure of the demand graph. We call a graph an *edge-K_{2m}-cycle* if it can be obtained from a cycle by replacing each edge of the cycle by an arbitrary number of parallel copies, and then subdividing some subset of these.

Lemma 3.7 *If a K_{2m} instance does not have an odd K_{2p} minor, and the demand graph contains an odd cycle, but no leaf demand edge, then the demand graph is either an edge-K_{2m}-cycle, or K_4.*

[1] A proof of this conjecture has recently been announced in [7].

Proof: Let C be the shortest odd cycle contained in the demand graph. We claim that each demand edge $v_i v_j$ must have at least one endpoint in $V(C)$. Otherwise, we can get an odd K_{2p} minor by contracting the supply edges incident to v_i and v_j to s, t respectively (and deleting any superfluous demands, or $v_{i'}$'s). As we have no leaf demand edges, we can assume any node v_i not in $V(C)$ is adjacent to at least two nodes of C in the demand graph.

First, suppose that C is a triangle. Then any node $v_i \notin V(C)$ adjacent to two nodes of C creates another triangle. It is easy to see that any two nodes $v_i, v_j \notin V(C)$ adjacent to two nodes of C must be adjacent to the same two nodes of C, otherwise we can find a triangle and an additional demand edge not connected to it; this leads to an odd K_{2p} minor as described above. Also, if $v_i, v_j \notin V(C)$ are adjacent to the same two nodes of C, neither of v_i, v_j is adjacent to the third node of C, because this would again yield an odd K_{2p} minor. So there are only two possibilities: Either the demand graph consists of many triangles all sharing the same edge of C, which is an edge-K_{2m}-cycle, or it is K_4.

Now, suppose C is an odd cycle of length at least 5. Suppose a node $v \notin V(C)$ is adjacent to two distinct nodes $v_1, v_2 \in V(C)$ in the demand graph. Then $v_1 v_2$ is not an edge of the demand graph for otherwise v along with v_1, v_2 would form a triangle in the demand graph contradicting the fact that C is the shortest odd cycle. Suppose the shortest path in C connecting v_1 and v_2 has length 3 or more. Then the edges vv_1 and vv_2 create a shortcut for C, which creates shorter cycles with each of the two paths in C from v_1 to v_2, one of them odd, contradicting the choice of C. Therefore, the shortest path in C connecting v_1 and v_2 must have length 2. This also implies that v has degree 2, because three nodes in C cannot all be at distance 2 of each other.

So any node $v \notin V(C)$ forms a bridge of length 2 between two nodes v_1 v_2 of C, with v_1 and v_2 being at distance 2 in C. Let v' be the middle node on the length 2 path between v_1 and v_2 in C. We claim that v' has degree 2 in the demand graph. To see this, note that the cycle C' obtained by replacing the path v_1, v', v_2 in C by the path v_1, v, v_2 has the same length as C and $v' \notin V(C')$. By the argument in the previous paragraph, v' has degree 2. It follows that v_1 and v_2 form a 2-cut for the demand graph, and all but one of the connected components obtained by removing v_1 and v_2 are singletons. As any node $v \notin C$ is adjacent to exactly two nodes v_1 and v_2 for which this is true, the demand graph is an edge-K_{2m}-cycle. ■

It is known that all demand graphs on less than five nodes are routable in any graph that satisfies the cut condition for it (see [30]), and hence K_4 is routable. So if we assume that a K_{2m} instance has no odd K_{2p} minor, but that no demand edge can be pushed, then the demand graph must be an edge-K_{2m}-cycle. We now show a property of the capacity and demand vectors of such instances.

Lemma 3.8 *Let (G, H) be a K_{2m} in which H is an edge-K_{2m}-cycle, no demand edge can be pushed, and any cut induced by a v_i is tight. Then for any node v that has degree 2 in both the demand and supply graph, the capacity and demand of the four edges incident to v in both graphs is the same.*

Proof: Let v be a node of degree 2 in the demand graph, connected to v_1 and v_2. By assumption, no edge of the demand graph can be pushed. So there are tight cuts S_1, T_1 separating vv_1 from s and t respectively. Similarly, there are tight cuts S_2, T_2 separating vv_2 from s and t respectively. By Lemma 3.6, the cuts S_1 and T_1 must both separate vv_1 from v_2. Similarly, S_2 and T_2 must separate vv_2 from v_1. Let us consider S_1 and T_1, and flip the side of v in both. The total capacity $|\delta_G(S_1)| + |\delta_G(T_1)|$ of both cuts is the same as before, but the total demand is modified by replacing the demand of vv_2 by that of vv_1. Since S_1 and T_1 were already tight, this means the demand of vv_1 is no greater than that of vv_2. Symmetrically, starting from S_2 and T_2, we prove that the demand of vv_2 is no greater than that of vv_1, and so the demand on both edges is the same. By repeating the argument on the pair of cuts S_1 and S_2, and the pair T_1 and T_2, we prove that the capacity of the edge sv is the same as that of vt. Since the cut induced by v is tight, the capacities and demands are all equal. ■

In the following, we call a node v *bracketed* by v_1 and v_2 if it is of degree 2 in the demand graph, and connected to v_1 and v_2. The previous lemma has the following easy corollary:

Corollary 3.9 *Let (G, H) be a K_{2m} in which H is an edge-K_{2m}-cycle, no demand edge can be pushed, and any cut induced by a v_i is tight. Let v be a node bracketed by v_1 and v_2. Let S be a tight cut separating s from* *t. Then either S separates v from both v_1 and v_2, or S separates v_1 from v_2, in which case the cut obtained by flipping the side of v in S is also tight.*

Proof: If S does not separate v_1 from v_2, then it must separate them from any node they bracket, because otherwise we get a tighter cut by flipping the bracketed node. If S does separate v_1 from v_2, then by Lemma 3.8, flipping v does not change the surplus of the cut. ■

The above corollary implies that for any tight cut separating s from t, there is another tight cut such that all nodes bracketed by a pair v_1 and v_2 are on the same side of the cut. Let us now consider any demand edge. If it cannot be pushed then there are tight cuts S and T separating the end points of the demand edge from s and t respectively. We flip all bracketed nodes so that they are all on the same side of S, and all on the same side of T. We can now consider all nodes bracketed by a same pair as a single node, since they are all on the same side of S and T. If we do so for all bracketed nodes, we reduce the demand graph to a cycle, which is routable, and hence contradicts the assumption that S and T are tight.

3.4 Compliant Instances

We define in this section the notion of compliant instance, and prove any such instance is routable if it is Eulerian and satisfies the cut condition. This is the main technical contribution of the paper. Recall that a demand edge e is called fully compliant if $G + e$ is series-parallel. If G is a series-parallel graph created from the edge st, we orient the edges of G by orienting the initial st and extending it naturally through series and parallel operations. We abuse notation and use G to refer to both the undirected graph and the oriented digraph.

In the resulting digraph, s is a unique source, and t is a unique sink; it is easy to see that this property is not lost by any series or parallel operation. The graph is also acyclic, because we can build an acyclic order starting from one for the st edge and extending it through the sequence of series and parallel operations. As a consequence, any directed path can be extended to an st path, because we can always add an edge at the beginning until it starts from s, and at the end until it ends at t.

We call a demand edge *compliant* if there is a directed path in G connecting its endpoints. We note that a compliant edge need not be fully compliant. For instance, if G consists of two node disjoint paths between s and t (essentially a cycle) then any demand edge uv is fully compliant but uv is not compliant if u, v are on the different paths and neither $\{u, v\} \cap \{s, t\} = \emptyset$. It is easy to show that if the s, t cut has three or more bridges, then in fact any fully compliant demand edge is also compliant. An instance is *compliant* if all edges are compliant or fully compliant.

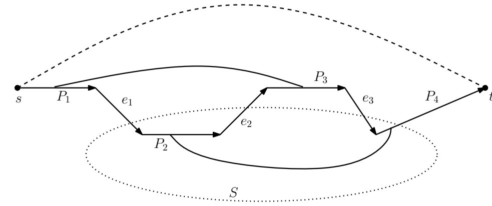
Theorem 3.10 *Let G be a series-parallel graph. Further let G, H be a compliant instance with $G + H$ Eulerian. If G, H satisfies the cut condition, then H has an integral routing in G.*

We start by two technical lemmas on oriented series-parallel graphs and on compliant demand edges.

Lemma 3.11 *Any directed path in G crosses at most twice the cut defined by a central set.*

Proof: Suppose there is a directed path P that crosses at least three times the cut $\delta(S)$ defined by a central set S. We extend P to an st path P', and denote by e_1, e_2 and e_3 the first three edges of P' crossing $\delta(S)$. P' can be decomposed as follows: $P' = (P_1, e_1, P_2, e_2, P_3, e_3, P_4)$ with P_1 starting from s and P_4 ending at t. Without loss of generality, we can assume P_1 and P_3 do not intersect S, P_2 is contained in S, and P_4 is fully or partially in S.

We show these four parts form a K_4, which is a contradiction. Indeed, P_1 and P_3 are connected by a path which does not cross $\delta(S)$, because S is central. Similarly, P_2 and P_4 are connected by a path contained in S. Since S is a series-parallel graph created from an st edge, adding an st edge should not create a K_4 minor. However, this would clearly be the case here, and so we have a contradiction. ■



Lemma 3.12 *Let G be a 2-connected series parallel graph obtained from an edge st. Let uv be a compliant edge that is not fully compliant. Then there is a directed s-t path P that contains u, v and in addition, there is a 2-cut l, r in G that separates u, v such that l, r lie on P. Moreover, if P traverses s, l, u, r, v, t in that order, then l, r can be chosen to separate u from both s and t (and symmetrically, if P traverses s, u, r, v, l, t we can separate v from s, t).*

Proof: Since uv is compliant, there is a directed path which without loss of generality traverses u and then v. This can be extended to a directed s-t path P traversing s, u, v, t in that order. Since uv is not fully compliant, there is by Lemma 3.2 a 2-cut l, r in G separating u from v. One node of the 2-cut, say r, has to be on the path from u to v. So P traverses $s - u - r - v - t$ in that order. Suppose l is not on P; then l, r separates s from t (otherwise, there would be a path $u - s - t - v$ in $G - \{l, r\}$). Since G is 2-node connected, there are two nodes-disjoint paths from s to t, one containing l and the other containing r. This implies that s, t is also a 2-cut separating l from r. If this were not the case, then $G - \{s, t\}$ would contain a path joining l, r. But then $G + st$ would clearly contain a K_4 minor, a contradiction. We now claim that s, r is a 2-cut that separates u, v, and hence choosing $l = s$ gives the desired cut. Suppose not, then there is a u-v path Q in $G - \{s, r\}$; this path Q necessarily has to contain l since l, r is a 2-cut for u, v. Let Q_1 be the sub-path of Q from u to l. If Q_1 contains t, then we have a $u - v$ path that avoids l, r, by following Q_1 and then the portion of P between t and v, a contradiction. Otherwise, Q_1 combined with the portion of P from u to r is an $l - r$ path that avoids s, t, again a contradiction. Hence we can assume that l also lies on P as claimed.

For the second part, assume that P traverses s, l, u, r, v, t in order. Clearly, there cannot be a path from u to t in $G' = G - \{l, r\}$ for otherwise it can be combined with P to find a $u - v$ path that avoids l, r. Suppose u is connected to $s \neq l$ in G'. Since G is 2-node-connected, there are two oriented nodes-disjoint s-t paths in G and one of them avoids l, r; otherwise one contains l and the other r. Hence these two paths along with the portion of P from l to r yield a K_4 in $G + st$. Thus s, t are connected in G' and hence if u can reach s in G' it can reach t, and hence v as well, again contradicting that l, r is a 2-cut for u, v. ■

We are now ready to prove Theorem 3.10.

Proof: Let uv be a demand edge which is compliant, but not fully compliant. We show that we can push uv into a series of fully compliant demand edges, maintaining the hypotheses for the new instance.

By Lemma 3.12 we have, without loss of generality, a directed s-t path P that traverses nodes s, l, u, r, v, t in that order (possibly $s = l$) where the component of $G - l, r$ containing u does not contain v, s or t.

We show that we can push the edge uv to l or r. Suppose this is not the case, then since l, r is a 2-cut, by Fact 2.2 there is a (central) tight cut $\delta(L)$ separating l from u, r and v, and a (central) tight cut $\delta(R)$ separating r from u, l and v. We show that this is not possible, by contracting the graph into a tri-source or a path-bipartite instance which again contains tight cuts corresponding to $\delta(L)$ and $\delta(R)$. Since we know that such instances are routable, these tight cuts could not exist.

Recall that $L \setminus R$ contains l but not r, and $R \setminus L$ contains r but not l. Hence we can contract $L \setminus R$ and $R \setminus L$ and label the nodes l' and r' respectively. (By Lemma 2.4, this is actually a contraction on connected subgraphs, although it is not critical at this point in the argument that we have a minor as such.) Denote the resulting instance (after removing loops) by G^*, H^*.

Since $\{l, r\}$ is a 2-cut separating u, v we have that the graph induced by the nodes $V \setminus (L \Delta R)$ has at least 2 components (one containing u and one containing v). Let C_i denote the components in this graph and let

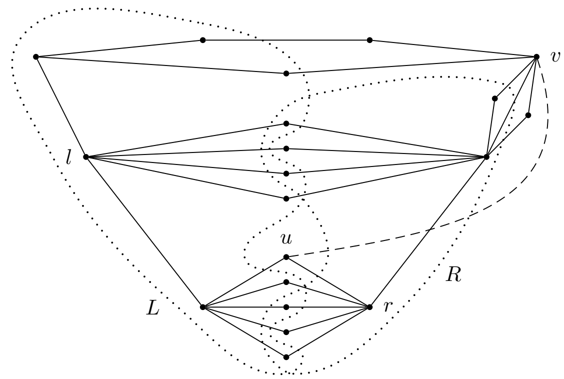


Figure 1: u and v are separated by the 2-cut l, r, and the demand uv cannot be pushed to l or r.

$X_i = R \cap L \cap C_i, Y_i = C_i \cap (V \setminus (R \cup L))$. Consider any component K of $G[Y_i]$. Since $G - L$ is connected, there is a path from K to $R \setminus L$ in the graph induced by $K \cup (R \setminus L)$. Similarly, there is a path from K to $L \setminus R$ in $G[K \cup (L \setminus R)]$, since $G - R$ is connected. Analogously, if K is a component of $G[X_i]$, then since $G[R]$ is connected, there must be a path from K to $R \setminus L$ in $G[K \cup (R \setminus L)]$. Similarly there is a path from K to $L \setminus R$. Now if both X_i, Y_i are nonempty, then we may choose a pair of components K, K' from X_i, Y_i respectively, and a path joining them in C_i, so that we can form $K_4 - e$ in the minor whose nodes are l', r', K, K'. This together with a path through some other C_j yields a K_4. Hence for each i, at most one of X_i or Y_i is nonempty. Moreover, if we shrink each C_i to a node c_i, then we have edges $c_i l', c_i r'$. The shrunken graph is therefore of type K_{2m}, with possibly an edge from l' to r'. Since each c_i was either of "type" $R \cap L$, or type $V \setminus (R \cup L)$, in the shrunken graph we can identify tight cuts induced by sets L', R' associated with our original pair L, R.

We next claim that neither s nor t is in R (and hence $R \setminus L$); to see this, note that the s-t path P goes successively through s, u, r, v, t and hence if either s or t is in R, then P would cross $\delta(R)$ three or more times.

We now examine two cases based on whether t is in L or not. In each case we examine the structure of the demand edges in the shrunken graph.

1. t is not in L. So t is contained in some C_i, say C_1. Suppose first that s lies in some C_i different from C_1. By Lemma 3.12, l, r separates u from s, t; so $u \notin C_i \cup C_1$. But then adding st (i.e., $c_i c_1$) to the shrunken graph (which maintains the series-parallel property), would create a K_4 on the nodes l', r', c_i, c_1, by considering the $l' - r'$ path through the component C_j containing u. Hence we may assume that s is either in L or in C_1.

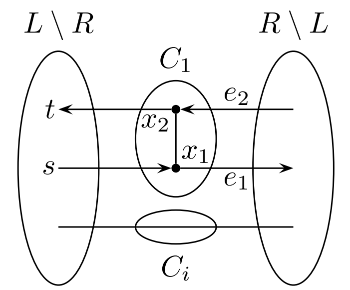
   Consider next some compliant demand edge $u'v'$ in the shrunken instance, and suppose that neither of its endpoints lie in C_1; say the endpoints are in C_i, C_j. Consider the directed s-t path P' associated with this demand where P' traverses s, u', v', t in that order. In the shrunken graph it must cross the cuts induced by C_1, C_i, C_j at least 5 times. Since every edge in any of these cuts lies in either $\delta(L')$ or $\delta(R')$, one of these two cuts is crossed three times, a contradiction. Hence, any compliant edge that remains in the shrunken graph, must have one endpoint in C_1. Thus the shrunken graph is a tri-source instance where l', r' and c_1 are the three sources.

2. t is in L. We claim that s is also in L; P goes successively through l, u and t, and hence if s is not in L, P would cross $\delta(L)$ three times.

   Recall that any directed path can be extended into a path starting at s and ending at t, and that this path should cross the cut of any central set at most twice by Lemma 3.11. We also use Lemma 2.4 to obtain that both $R \setminus L, L \setminus R$ are central sets. Thus since s and t are both in $L \setminus R$, there are two types of directed s-t paths. The first type does not ever enter $R \setminus L$. If such a path ever leaves $L \setminus R$, then it must enter some C_i. It must then leave C_i to reach t, and hence it enters $L \setminus R$ again. Thus the path has crossed the cut of $L \setminus R$ twice, and hence it can not leave it again. The second type of path may traverse $R \setminus L$. This is the only type that can traverse more than one C_i. We claim that a directed s-t path of the second type goes from $L \setminus R$ to $R \setminus L$, traversing at most one C_i on the way, then goes back from $R \setminus L$ to $L \setminus R$, traversing at most one C_i on the way. This follows since any s-t directed path cannot cross $\delta(R \setminus L)$ more than twice, so it can enter at most once, and leave at most once.

Therefore, a directed s-t path of the second type leaves $L \setminus R$, possibly traverses a C_i, enters $R \setminus L$, leaves $R \setminus L$, possibly traverses a C_i, and enters $L \setminus R$ in that order. It cannot leave $L \setminus R$ again after, as it has crossed its cut twice. As a consequence, no directed path can traverse more than two C_i's.

We now show that for any C_i, the arcs (oriented edges of G) connecting C_i to $R \setminus L$ are all oriented in the same direction. Assume there is a C_i, say C_1, with an arc e_1 entering and an arc e_2 leaving $R \setminus L$. We can extend e_1 into a directed path P_1 starting with s. Since P_1 ends by entering $R \setminus L$ with e_1, it did not leave $R \setminus L$ before, so it entered C_1 from $L \setminus R$. We can also extend e_2 into a directed path P_2 ending at t. Since P_2 starts by leaving $R \setminus L$ with e_2, it can not enter $R \setminus L$ again, so it leaves C_1 for $L \setminus R$.



We now show P_1 and P_2 are node-disjoint inside C_1. Suppose that it is not the case, and that there is some $x \in C_1$ contained both in P_1 and P_2. Then we can create a directed path leaving $R \setminus L$ then entering it again, by following P_2 from e_2 to x, then following P_1 from x to e_1. The directed path is simple, since the graph is acyclic. We can extend it to a directed s-t path which then contradicts our previous claims. Hence P_1 and P_2 are node-disjoint inside C_1. But since C_1 is connected, these paths are connected by some path. That is, for any $x_1 \in C_1$ on P_1 and any $x_2 \in C_1$ on P_2, there is a path inside C_1 connecting x_1 to x_2. Also, since u and v are not in the same C_i, there is at least one other C_i linking $L \setminus R$ and $R \setminus L$. This now creates a K_4 in the graph (with l', r' shrunken) on the nodes l', r', x_1, x_2, a contradiction.

Thus for any C_i, the arcs connecting C_i to $R \setminus L$ are all oriented in the same direction. Therefore the C_i's are partitioned into two types: *out* components are those with arcs going into $R \setminus L$, and *in* components with arcs entering from $R \setminus L$. It follows that any directed path traversing two C_i's traverses one of each type. Any compliant edge $u'v'$ that remains in the shrunken graph that is not incident to l' or r', must admit a directed $u'v'$ with its endpoints in distinct components. Hence exactly one of u', v' lies in an in component, and the other lies in an out component. Thus the shrunken graph is a path-bipartite instance.

In either case, the instance G^*, H^* obtained is routable by Lemma 3.4. Therefore, the cuts $\delta(L)$ and $\delta(R)$ could not have both been tight, and so we can push uv to either l or r. By induction, we can push any compliant edge into a series of fully compliant edges. ■

4 Integral Routing with Congestion

4.1 Congestion 5 Routing in Series Parallel Graphs

Theorem 4.1 *Suppose G, H is a series-parallel instance satisfying the cut condition. Then H has an integral routing with edge congestion* 5.

Proof: By the result of [21], there is a fractional routing f of H with congestion 2. For any demand edge xy, suppose that s', t' induce the highest level (with respect to the decomposition of G starting from st) 2-cut separating x, y. Then at least half of any fractional flow for xy has to go either via s' or t'. Without loss of generality, assume it is s'. We push the xy demand edge to s', i.e., we create demand edges xs', ys' and remove xy. We do this simultaneously for all demand edges - we are pushing demands based on the fractional flow

f. The new instance is compliant. Let us call H' the new demand graph. By construction, there is a feasible fractional flow of $1/2$ of each demand from H' in $2G$. This implies that $4G$ satisfies the cut condition for H'. In order to make the graph Eulerian, we can add a T-join, where T is the set of odd degree nodes in $4G + H'$. Since we can assume that G is connected by previous reductions, we may choose such a T-join as a subset of $E(G)$. It follows that we can create an Eulerian, compliant instance G', H' that satisfies the cut condition, and G' is a subgraph of $5G$. Hence by Theorem 3.10, we may integrally route H', and hence H in the graph $5G$. ■

We believe that the above result can be strengthened substantially and postulate the following:

Conjecture 4.2 *Let G, H satisfy the cut condition where G is series-parallel and $G + H$ is Eulerian. Then there is a congestion 2 integral routing for H.*

4.2 Rerouting Lemma from [10]

We state the rerouting lemma that we referred to in the introduction. It is useful to refer to the informal version we described earlier. Let D be a demand matrix in a graph G and let $f : V \to V$ be a mapping. We define a demand matrix D_f as follows:

$$D_f(xy) = \sum_{uv: f(u)=x, f(v)=y} D(uv).$$

In other words the demand $D(uv)$ for a pair of nodes uv is transferred in D_f to the pair $f(u)f(v)$. Thus the total demand transferred from u to $f(u)$ is $\sum_v D(uv)$. We define another demand matrix D'_f which essentially asks that each node u can send this amount of flow to $f(u)$.

$$D'_f(uf(u)) = \sum_v D(uv).$$

Proposition 4.3 *If D'_f is (integrally) routable in G with congestion a, and D_f is (integrally) routable in G with congestion b, then D is (integrally) routable with congestion $a + b$ in G.*

We need a cut condition given by the simple lemma below. For completeness, we include a proof in the appendix (A.1).

Lemma 4.4 ([10]) *Let D be a demand matrix on a given graph G and let $f : V \to V$ be a mapping. If the cut condition is satisfied for D, and D'_f is routable in γG, then the cut condition is satisfied for D_f in $(\gamma + 1) \cdot G$.*

We give a useful corollary of the above lemma.

Corollary 4.5 *Let $G = (V, E)$ satisfy the cut-condition for $H = (V, E_H)$ and let $A \subseteq V$ be a node-cover in H. Then there exists a demand graph $I = (A, F)$ such that $2G$ satisfies the cut-condition for I. Moreover, if I is (integrally) routable in $2G$ with congestion α, then H is (integrally) routable in G with congestion $(1 + 2\alpha)$.*

Proof: Assume for simplicity that G and H have unit capacities. Let $A \subset V$ be a node-cover in H. Shrink A to a node a to obtain a new supply graph G' and a new demand graph H'. Since A is a node-cover in H, all demand edges in H' are incident to a; therefore H' is a star and G', H' is a single-source instance. For simplicity assume that there are no parallel edges in H'; if node u has $d > 1$ parallel edges to a, then add d dummy terminals connected to u with infinite capacity edges in G' and replace each (u, a) edge by an edge from a dummy terminal to a. Let $S \subseteq V \setminus A$ be the set of nodes that have a demand edge to a in H'. Note that G' satisfies the cut-condition for H'. Therefore by the maxflow-mincut theorem (or Menger's theorem) H' is routable in G' with congestion 1 and by our assumption that the demands are unit valued and capacities are integer valued, the flow corresponds to $|S|$ paths, one from each node in S to a. Now unshrink a to A; thus the flow corresponds to paths from S to A in G. Define a mapping $f : V \to V$ where $f(u) = u$ if $u \in V \setminus S$ (we only care about $u \in A$), and if $u \in S$ then $f(u) = v$ where v is the end point in A of the path from u to A. Let D be the demand matrix corresponding to H and D_f be the demand matrix induced by the mapping f (recall that this the matrix where each demand uv is transferred to $f(u)f(v)$). Let $I = (A, F)$ be the demand graph induced by f. Recall that D'_f is the demand matrix where $D'_f(uf(u)) = \sum_v D(uv)$; note then that D'_f corresponds to the single-sink flow problem determined by H'. Hence by Lemma 4.4, D_f, and hence I, satisfies the cut condition in $2G$. We then apply Proposition 4.3 to see that if I is (integrally) routable in $2G$ with congestion α (which is the same as I being routable in G with congestion 2α), then H is (integrally) routable in G with congestion $(1 + 2\alpha)$ since H' is integrally routable in G with congestion 1. ■

4.3 k-Outerplanar and k-shell Instances

Let $G = (V, E)$ be an embedded planar graph. We define the outer layer or the 1-layer of G to be the nodes of G that are on the outer face of G. The k-th layer of G is the set of nodes of V that are on the outer face of G after the nodes in the first $k - 1$ layers have been removed. A *k-Outerplanar* graph is a planar graph that can be embedded with at most k layers. We let V_i denote the nodes on the i-th layer. We are interested in multiflows in planar graphs when at least one terminal of each demand pair lies in one of the outer k layers; we call such instances *k-shell instances*. Let $H = (V, F)$ be a demand graph.

Theorem 4.6 (Okamura-Seymour [25]) *Let G be a planar graph and H be a demand graph with all terminals on a single face. If H satisfies the cut-condition, then there is a half-integral routing of H in G. Moreover if $G + H$ is Eulerian, H is integrally routable in G.*

Theorem 4.7 ([8]) *If G is a k-Outerplanar graph and H satisfies the cut-condition, then H is fractionally routable in G with congestion c^k for some universal constant c.*

We can strengthen the above theorem to prove the following on k-shell instances.

Theorem 4.8 *Let G be a planar graph and let $V' = \cup_{i=1}^k V_i$ be the set of nodes in the outer k layers of a planar embedding of G. Suppose $H = (V, F)$ is a demand graph where for each demand edge at least one of the end points is in V'. If G, H satisfy the cut condition, then H can be fractionally routed in G with congestion c^k for some universal constant c.*

The proof of Theorem 4.7 relies on machinery from metric embeddings. Our proof of Theorem 4.8 also relies on metric embeddings, and in particular uses recent results [22] and [13]. Since these techniques are somewhat orthogonal to the primal methods that we use in this paper, we describe a proof of Theorem 4.8 in a separate manuscript [12]. Below we relate the integer and fractional flow-cut gaps for k-shell instances.

Theorem 4.9 *Let G be a planar graph and let $V' = \cup_{i=1}^k V_i$ be the set of nodes in the outer k layers of a planar embedding of G. Suppose $H = (V, F)$ is a demand graph where for each demand edge at least one of the end points is in V'. If H is fractionally routable in G, then it can be integrally routed in G with congestion 6^k.*

Combining the above two theorems, we have:

Corollary 4.10 *If G, H is a k-shell instance that satisfies the cut-condition, then H is* integrally *routable in G with congestion c^k for some universal constant c.*

We need the following claim as a base case to prove Theorem 4.9.

Claim 4.11 *Let G be a planar graph and $H = (V, F)$ be a demand graph such that for each demand edge at least one end point is on the outer face V_1. If G, H satisfy the cut-condition, then there is an integral routing of H in G with congestion 5.*

Proof: We observe that V_1 is a node cover in $H = (V, F)$. Therefore, by Corollary 4.5 there is a demand graph $I = (V_1, F')$ such that $2G, I$ satisfy the cut condition. By the Okamura-Seymour theorem (Theorem 4.6), I is integrally routable in $2G$ with congestion 2. Therefore, by Corollary 4.5, H is integrally routable in G with congestion 5. ∎

Proof:[of Theorem 4.9] We prove the theorem by induction on k. The base case of $k = 1$ follows from Claim 4.11.

Assuming the hypothesis for $j < k$, we prove it for $j = k$. Let $H = (V, F)$ be a demand graph that is fractionally routable in G and such that each demand edge is incident to a node in the outer k layers. Let $H_k = (V, F_k)$ be the subgraph of H induced by the demand edges $F_k \subseteq F$ that are incident to at least one node in V_k and moreover the other end point is not in $V_1 \cup \ldots \cup V_{k-1}$. We obtain a new supply graph G' by shrinking the nodes in $\cup_{i=1}^{k-1} V_i$ to a single node v. Note that H_k is fractionally routable in G' as well. Fix some arbitrary routing of H_k in G'. Partition F_k into F_k^a and F_k^b as follows. F_k^a is the set of all demands that route at least half their flow through v in G'. $F_k^b = F_k \setminus F_k^a$. Thus a demand in F_k^b routes at least half its flow in the graph $G'' = G[V \setminus \cup_{i=1}^{k-1} V_i]$. We claim that the demand graph $H_k^b = (V \setminus \cup_{i=1}^{k-1} V_i, F_k^b)$ is integrally routable in G'' with congestion 10. For this we note that H_k^b is fractionally routable in $2G''$ which implies that $(2G'', H_k^b)$ satisfies the cut-condition. Moreover V_k is the outer-face of G'' and each demand in F_k^b has at least one end point in V_k and hence we can apply Claim 4.11.

Now consider demands in F_k^a and their flow in G'. For simplicity assume that the end points of F_k^a are disjoint and let T be the set of end points. By doubling the fractional flow in G' of each demand $f \in F_k^a$ we get a feasible routing for sending one unit of flow from each $t \in T$ to v. Thus in G' there are paths $P_t, t \in T$ where P_t is a path from t to v and no edge has more than two paths using it. These paths imply that the terminals in T can be integrally routed to nodes in $\cup_{i=1}^{k-1} V_i$ with congestion 2 in G (simply unshrink v). For each demand $f = (u, v) \in F_k^a$ let $f' = (u', v')$ be a new demand where u' and v' are the nodes in $\cup_{i=1}^{k-1} V_i$ that u and v are routed to. Let these demands be F_k^c. We claim that F_k^c is fractionally routable in G with congestion 3 - simply concatenate the routing of F_k^a with the paths that generated F_k^c from F_k^a. Now consider the demand graph $H' = (V, (F \setminus F_k) \cup F_k^c)$. Since $H = (V, F)$ is fractionally routable in G and F_k^c is fractionally routable in $3G$, we have that H' is fractionally routable in $4G$. Also, each demand in H' has an end point in $\cup_{i=1}^{k-1} V_i$. Therefore, by the induction hypothesis, H' is integrally routable in G with congestion $4 \cdot 6^{k-1}$.

Routing H as above consists of routing H', the routing of F_k^a, and the routing of the demands in F_k^b to the outer $k-1$ layers; adding up the congestion for each of these routings as shown above, we see that H is routable in G with congestion $4 \cdot 6^{k-1} + 10 + 2 \leq 6^k$ for $k \geq 2$. This proves the hypothesis for k. ∎

4.4 Flow-Cut Gap and Node Cover size of Demand Graph

Linial, London and Rabinovich [23] and Aumann and Rabani [3] showed that if the supply graph $G = (V, E)$ satisfies the cut condition for a demand graph $H = (V, E_H)$, then H is routable in G with congestion $O(\log k)$ where $k = |E_H|$; to obtain this refined result (instead of an $O(\log n)$ bound), [23, 3] rely on Bourgain's proof of the distortion required to embed a finite metric into ℓ_1. Günlük [14] further refined the bound and showed that the flow-cut gap is $O(\log k^*)$ where k^* is the size of the smallest node cover in H; recall a *node cover* is a subset S of nodes for which every edge of H has at least one endpoint in S. For example if $k^* = 1$, then H induces a single-source problem for which the flow-cut gap is 1. Günlük's argument requires a fair amount of technical reworking of Bourgain's proof. Here we give a simple and insightful proof via Lemma 4.4, in particular Corollary 4.5.

Theorem 4.12 *Let $G = (V, E)$ satisfy the cut-condition for $H = (V, E_H)$ such that H has a node-cover of size k^*. Then H is routable in G with congestion $O(\log k^*)$.*

Proof: Let $A \subset V$ be a node-cover in H such that $|A| = k^*$. We now apply Corollary 4.5 which implies that there is a demand graph $I = (A, F)$ such that $2G$ satisfies the cut-condition for I. Moreover if I is routable in

$2G$ with congestion α then H is routable in G with congestion $(1 + 2\alpha)$. Note that I is a demand graph with at most $(k^*)^2$ edges, therefore, it is routable in $2G$ with congestion $O(\log k^*)$ [23, 3]. Hence H is routable in G with congestion $O(\log k^*)$. ■

4.5 Multiflows with terminals on k faces of a planar graph

Lee and Sidiropoulos [22] recently gave a powerful methodology via their *peeling* lemma to reduce the flow-cut gap question for a class of instances to other potentially simpler class of instances. Using this they reduced the flow-cut gap question for minor-free graphs to planar graphs and graphs closed under bounded clique sums. One of the applications of their peeling lemma is the following result. Let G be an embedded planar graph and H be a demand graph such that the endpoints of edges in H lie on at most k faces of G. If G satisfies the cut-condition, then H is (fractionally) routable in G with congestion $e^{O(k)}$. Their proof extends to graphs of bounded genus and relies on the non-trivial peeling lemma. Here we give a simple proof with a stronger guarantee for the planar case, again using Lemma 4.4.

Theorem 4.13 *Let $G = (V, E)$ be an embedded planar graph and $H = (V, F)$ be a demand graph such that the endpoints of edges in H lie on at most k faces of G. If G satisfies the cut-condition, then H is routable in G with congestion $3k$. Moreover if G and H have integer capacities and demands respectively, then there is an integral flow with congestion $5k$.*

Proof: Let $V_1, \ldots, V_k$ be the node sets of the k faces on which the demand edges are incident to. Let $F_i \subseteq F$ be the edges in H that have at least one end point incident to a node in V_i and let $H_i = (V, F_i)$ be the demand graph induced by F_i. Note that G satisfies the cut-condition for H_i. Clearly V_i is a node-cover for $H_i = (V, F_i)$ and hence by Corollary 4.5, there is a demand graph $I_i = (V_i, F'_i)$ such that $2G$ satisfies the cut-condition for I_i. Note, however, that I_i is an Okamura-Seymour instance in that all terminals lie on a single face. Hence I_i is routable in $2G$ with congestion 1 and is integrally routable in $2G$ with congestion 2. Hence, by Corollary 4.5, H_i is routable in G with congestion 3 and integrally routable in G with congestion 5. By considering $H_1, \ldots, H_k$ separately, H is routable in G with congestion $3k$ and integrally with congestion $5k$. ■

Remark: We observe that a bound of k is easy via the Okamura-Seymour theorem if for each demand edge the two end points are incident to the same face. The rerouting lemma allows us to easily handle the case when the end points may be on different faces; in fact the proof extends to the case when nodes on k faces in G form a node cover for the demand graph H. The above proof can be easily extended to graphs embedded on a surface of genus g to show a flow-cut gap of $3\alpha_g k$ where α_g is the gap for instances in which all terminals are on a single face.

5 Lower-bound on flow-cut gap in series-parallel graphs

Lee and Raghavendra have shown in [21] that the flow-cut gap in series-parallel graph can be arbitrarily close to 2; this lower bound matches an upper bound found previously [6]. They introduce a family of supply graphs and prove the required congestion using the theory of metric embeddings. In particular, their lower bound is shown by a construction of a series parallel graph and a lower bound on the distortion required to embed the shortest path metric on the nodes of the graph into ℓ_1.

In this section we give a different proof of the lower bound. We use the same class of graphs and the recursive construction of Lee and Raghavendra [21]. However, we use a primal and direct approach to proving the lower bound by constructing demand graphs that satisfy the cut-condition but cannot be routed. We lower bound the congestion required for the instances by exhibiting a feasible solution to the dual of the linear program for the maximum concurrent multicommodity flow in a given instance. This is captured by the standard lemma below which follows easily from LP duality.

Lemma 5.1 *Let $G = (V, E)$ and $H = (V, F)$ be undirected supply and demand graphs respectively with c_e denoting the capacity of $e \in E$ and d_f denoting the demand of $f \in F$. Then the minimum congestion r required to route H in G is given by*

$$\min_{\ell : E \to \mathcal{R}^+} \frac{\sum_{f \in F} d_f \ell_f}{\sum_{e \in E} c_e \ell_e}$$

where ℓ_f is the shortest path distance between the end points of f in G according to the non-negative length function $\ell : E \to \mathcal{R}^+$.

Corollary 5.2 *By setting $\ell_e = 1$ for each $e \in E$, the minimum congestion required to route H in G is at least D/C where $C = \sum_{e \in E} c_e$, and $D = \sum_{f \in F} d_f \ell_f$, where ℓ_f is the shortest path distance in G between the end points of f (with unit-lengths on the edges).*

We refer to C and D as defined in the corollary above as the total capacity and the total demand length respectively. We refer to the lower bound D/C as the *standard* lower bound.

Construction of the Lower Bound Instances: The family of graphs presented in [21] is obtained from a single edge by a simple operation, which consists in replacing every edge by a $K_{2,m}$ graph, as shown in Figure 2.

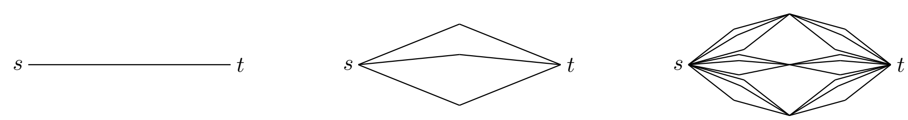


Figure 2: Supply graph for gap example. Replace each edge by $K_{2,m}$, here $m = 3$.

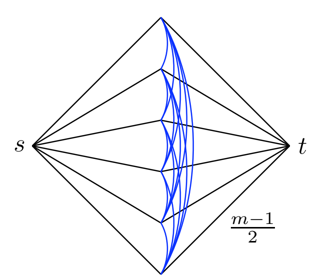


Figure 3: Building block for gap example.

Assuming m is even, consider the following demands and capacities for a $K_{2,m}$ graph: a complete demand graph on the m nodes of degree 2, with demand 1 on each edge, and a capacity $(m-1)/2$ on each supply edge. See Figure 3. One easily checks that the cut condition holds for this instance. Moreover, the total capacity is $2m \cdot (m-1)/2 = m(m-1)$, and the total demand length is $2 \cdot m(m-1)/2 = m(m-1)$; hence the standard lower bound $D/C = 1$, and in fact the instance is routable. The simple yet important observation is that for any central cut separating s from t, the surplus of the cut is equal to $m(m-1)/2 - k(m-k)$, when the cut separates the degree 2 nodes into sets of cardinality k and $m-k$. The minimum surplus is attained when $k = m/2$, where it is $m(m-2)/4$. We call this instance $I(s,t)$. Based on this we define an instance $I_c(s,t)$ for any real $c > 0$ as the instance obtained from $I(s,t)$ by multiplying all demands and capacities of $I(s,t)$ by $\frac{c}{m(m-2)/4}$. The effect of this is that $I_c(s,t)$ satisfies the cut condition, has a standard lower bound of 1 and is routable, and the surplus of any central cut separating s and t is at least c.

We summarize the properties of $I_c(s,t)$ in the following lemma, with the last property being a crucial one.

Lemma 5.3 *The instance $I_c(s,t)$ has the following properties:*

1. *It has $2m$ supply edges, each of capacity $\frac{c}{m(m-2)/4}\frac{m-1}{2}$, and it has $\frac{m(m-1)}{2}$ demand edges, each with demand $\frac{c}{m(m-2)/4}$.*
2. *The standard lower bound for $I_c(s,t)$ is 1 and the instance is routable.*
3. *The minimum surplus of any cut separating s from t is equal to c.*
4. *Adding an st demand edge of demand c to $I_c(s,t)$ preserves the cut condition but the standard lower bound for this modified instance is $(1+\frac{m-2}{2(m-1)})$.*

Proof: The first three facts follow directly from the description of $I_c(s,t)$. The surplus of any st cut in $I_c(s,t)$ is c, and $I_c(s,t)$ satisfies the cut condition. Hence, it follows that adding an st demand edges with demand c preserves the cut condition. We note that for $I_c(s,t)$, the total capacity $C = 2m \cdot \frac{c}{m(m-2)/4}\frac{m-1}{2} = \frac{4c(m-1)}{(m-2)}$, this is also equal to the total demand length D for $I_c(s,t)$. Adding an st demand edge with demand value c increases the total demand length by $2c$, and therefore the standard lower bound for the modified instance is $\frac{C+2c}{C} = (1+\frac{m-2}{2(m-1)})$. ■

As in [21], the goal is to show that the congestion lower bound obtained in the construction above can be amplified by iteration. As a first step, the next lemma shows that replacing an edge of the supply graph by an instance of type $I_c(s,t)$ maintains the cut condition. Given a multiflow instance G, H and a supply edge uv in G we can obtain a new instance G', H' by replacing uv by the instance $I_c(s,t)$ where we identify u with s and t with v. Note that all H' contains all the demand edges of H and the demand edges of $I_c(s,t)$.

Lemma 5.4 *Let $I = G, H$ be a multiflow instance that verifies the cut condition. Let uv be a supply edge in G with capacity $c > 0$. Let $I' = G', H'$ be a new instance obtained by replacing uv by $I_c(s,t)$. Then G', H' verifies the cut condition.*

Proof: The surplus of any cut separating u from v in I' is at least as big as the surplus of the corresponding cut in I, since the surplus of any cut separating s from t in $I_c(s,t)$ is at least c. Also, the surplus of any cut in I' that does not separate u from v is equal to the surplus of the corresponding cut in I, plus the nonnegative surplus of some cut in $I_c(u,v)$. ■

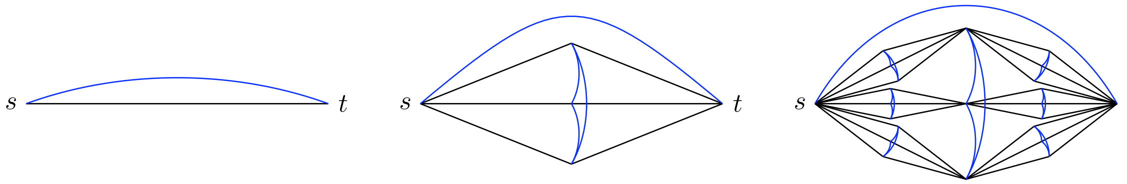
¿From any instance, we build an instance requiring a larger congestion using the following transformation: We replace *each* supply edge by an instance of type $I_c(u,v)$. The next theorem gives a lower bound on the required congestion in the transformed instance.

Theorem 5.5 *Let $I = G, H$ be an instance verifying the cut condition, with total capacity C and total demand length D. Let I' be the instance obtained by replacing each supply edge uv of capacity c_{uv} by an instance $I_{c_{uv}}(u,v)$. Then the transformed instance $I' = G', H'$ verifies the cut condition, and the standard lower bound for I' is $1 + \frac{D}{C}\cdot\frac{m-2}{2(m-1)}$.*

Proof: The fact that I' verifies the cut condition is a direct consequence of Lemma 5.4.

Let us first compute the total capacity of G'. Since each supply edge of capacity c in I is replaced with $2m$ supply edges of capacity $\frac{c}{m(m-2)/4}\frac{m-1}{2}$, the total capacity of G' is $C' = C\frac{2m}{m(m-2)/4}\frac{m-1}{2} = C\frac{4(m-1)}{m-2}$.

The demand edges in H' either exist in H, or are added by the transformation, i.e. they are internal to some $I_c(u,v)$ instance that replaces a supply edge uv. The total demand length in I' can therefore be decomposed into the part corresponding to the demand edges that exist in I, and the total demands internal to each $I_c(u,v)$ instance. In each $I_c(u,v)$ instance, the total demand length is equal to the total capacity. So the sum of demand lengths internal to each $I_c(u,v)$ instance is equal to the total capacity, which is the total capacity of G'. On the



$r = 1$ $r = f(1)$ $r = f(f(1))$

Figure 4: The flow-cur gap r grows bigger with each iteration.

other hand, the demand edges that exist in I have the same demand, but the shortest path of each such edge has exactly doubled in the transformed instance. It follows that $D' = C' + 2D$.

Therefore, the standard lower bound for I' is is $\frac{D'}{C'} = \frac{C'+2D}{C'} = 1 + \frac{2D}{C}\frac{m-2}{4(m-1)} = 1 + \frac{D}{C}\frac{m-2}{2(m-1)}$. ■

Thus, Theorem 5.5 can be used to amplify the flow-cut gap. In particular, if the standard lower bound yields a gap of x for an instance I then, one obtains an instance with standard lower bound yielding a gap of $f(x) = 1 + x\frac{m-2}{2(m-1)}$. We can iterate this process k times yielding instances with flow-cut gap $f^{(k)}(x)$ where $f^{(k)}$ is the function f composed k times. We note that

$$f^{(k)}(x) = 1 + \frac{(m-2)}{2(m-1)} + \cdots + \left(\frac{(m-2)}{2(m-1)}\right)^{k-1} + x\left(\frac{(m-2)}{2(m-1)}\right)^{k},$$

and that this converges to

$$\lim_{k\to\infty} f^{(k)}(x) = 2 - \frac{2}{m},$$

for any x.

Theorem 5.6 *For any $\varepsilon > 0$, there is a series-parallel graph instance for which the flow-cut gap is $2 - \varepsilon$.*

Proof: We apply the iterated construction starting with the instance consisting of a single edge of capacity 1 as the supply graph, and a demand graph consisting of the same edge with demand 1; clearly, the standard lower bound for this instance is 1. See Figure 4. We observe that the construction preserves the property that the supply graphs are series parallel. Thus, if we set $m = \lceil 4/\varepsilon \rceil$ in $I(s,t)$, the iterated construction yields a series parallel graph instance with flow-cut gap arbitrarily close to $2 - \varepsilon/2$, and hence one can choose a sufficiently large k such that iterating the construction k times gives an instance with gap at least $2 - \varepsilon$. ■

Acknowledgements: This work was partly done while the first two authors were at Lucent Bell Labs; they acknowledge support from an ONR grant N00014-05-1-0256. C. Chekuri is also partly supported by NSF grant CCF-0728782. We thank Anupam Gupta and Sanjeev Khanna for several discussions and ideas along the course of this work. We thank an anonymous reviewer for pointing out a minor error in an earlier proof of Theorem 4.9. We thank an anonymous reviewer of the journal version for several helpful comments.

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A Appendix

A.1 Proof of the Rerouting Lemma [10]

For completeness we give the proof from [10].

Proof: Consider a cut $\delta_G(S)$ in G for some $S \subset V$. Let $D(S)$ denote the total demand across this cut. Since D satisfies the cut condition $|\delta_G(S)| \geq D(S)$ for all $S \subset V$. Also, since D'_f is routable in γG, $\gamma|\delta_G(S)| \geq D'_f(S)$ for all $S \subset V$.

To prove the lemma we need to show that $(\gamma+1)|\delta_G(S)| \geq D_f(S)$. From the above inequalities, it suffices to show that $D_f(S) \leq D(S) + D'_f(S)$. Let X_S denote the set of all unordered pairs of nodes uv such that u and v are separated by S, that is $|\{u,v\} \cap S| = 1$. We can write $D_f(S)$ as $\sum_{uv:f(u)f(v)\in X_S} D(uv)$. For each pair uv such that $f(u)f(v) \in X_S$, we charge $D(uv)$ to either $D(S)$ or $D'_f(S)$ such that there is no overcharge. This will complete the argument.

We consider two cases. If $uv \in X_S$ then we charge $D(uv)$ to $D(S)$. Note that $\sum_{uv\in X_S} D(uv) = D(S)$ and hence we do not over charge $D(S)$. If $uv \notin X_S$, then either $uf(u) \in X_S$ or $vf(v) \in X_S$ but not both. In $uf(u) \in X_S$ we charge $D(uv)$ to u, otherwise to v. We observe that the total charge to a node u is at most $D'_f(uf(u))$ and it is charged only if $uf(u) \in X_S$. Hence the total charge to $D'_f(S)$ is not exceeded either. ■