A note on the hardness of approximating the
**K-way Hypergraph Cut** problem

Chandra Chekuri∗  Shi Li†

November 22, 2015

Abstract

We consider the approximability of **K-way Hypergraph Cut** problem: the input is an edge-weighted hypergraph $G = (V, E)$ and an integer $k$ and the goal is to remove a min-weight subset of the edges such that the residual graph has at least $k$ connected components. When $G$ is a graph this problem admits a $2(1 - 1/k)$-approximation [8], however, there has been no non-trivial approximation ratio for general hypergraphs. In this note we show, via a very simple reduction, that an $\alpha$-approximation for **K-way Hypergraph Cut** implies an $O(\alpha^2)$-approximation for the **Densest k-Subgraph** problem. This gives conditional hardness of approximation for **K-way Hypergraph Cut** since the best known approximation ratio for **Densest k-Subgraph** is $O(n^{1/4+\epsilon})$ [1] and resolving its approximability is a major open problem. As a corollary we obtain conditional hardness for **k-way Submodular Multiway Partition** problem which generalizes **K-way Hypergraph Cut**. These resuls are in contrast to $2(1 - 1/k)$-approximation for closely related problems where the goal is to separate $k$ given terminals [3, 4].

1 Introduction

We consider the following problem.

**K-way Hypergraph Cut**: Let $G = (V, E)$ be hypergraph with edge weights given by $w : E \to \mathbb{R}_+$. Given an integer $k$, find a min-weight subset of edges $E' \subseteq E$ such that $G - E'$ has at least $k$ connected components. Equivalently find a partition of $V$ into $k$ non-empty sets $V_1, V_2, \ldots, V_k$ such that the weight of the hyperedges that cross the partition is minimized.

**K-way Hypergraph Cut** is known as the **k-Cut** problem when the input is a graph and is one of the well-studied variants of graph partitioning problems. **K-way Hypergraph Cut** is a special case of a more general submodular partitioning problem defined below.

**k-Way Submodular Partition (k-way Sub-MP)**: Let $f : 2^V \to \mathbb{R}_+$ be a non-negative submodular set function over a finite ground set $V$. The $k$-way submodular partition problem is to find a partition $V_1, \ldots, V_k$ of $V$ to minimize $\sum_{i=1}^{k} f(V_i)$ such that for $1 \leq i \leq k$, $V_i \neq \emptyset$. An important special case is when $f$ is symmetric and we refer to it as **k-way Sym-Sub-MP**.

---

*Dept. of Computer Science, University of Illinois, Urbana, IL 61801. Supported in part by NSF grants CCF-1319376 and CCF-1526799. chekuri@illinois.edu
†Dept. of Computer Science and Engineering, University at Buffalo, Buffalo, NY 14260. shil@buffalo.edu

1A hyperedge $e$ crosses a partition of the vertex set if $e$ properly intersects at least two parts of the partition.

2A set function $f : 2^V \to \mathbb{R}$ is submodular iff $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ for all $A, B \subseteq V$. Moreover, $f$ is symmetric if $f(A) = f(V - A)$ for all $A \subseteq V$. 
We refer the reader to [3] to see why K-way Hypergraph Cut is a special case of K-way SUB-MP. The K-CUT problem is not only a special case of K-way Hypergraph Cut but it is also a special case of K-way SYM-SUB-MP. When \( k \) is part of the input all the problems we discussed so far are NP-Hard, and also APX-Hard to approximate. K-way SYM-SUB-MP admits a \( 2(1 - 1/k) \)-approximation [7, 11] and hence also K-CUT [8]. For K-way Hypergraph Cut a \( 2\Delta(1 - 1/k) \)-approximation easily follows from the \( 2(1 - 1/k) \)-approximation for K-CUT; here \( \Delta \) is the rank of the hypergraph (the maximum size of any hyperedge). On the other hand, in the general case, the known approximation algorithms for K-way Hypergraph Cut and K-way SUB-MP provide an approximation ratio of \( \Omega(k) \) [11]. It was an open problem to obtain an improved understanding of their approximability.

In this note we show that a good approximation for K-way Hypergraph Cut would imply a good approximation for the Densest k-Subgraph problem. Resolving the approximability of Densest k-Subgraph is a well-known open problem. We first describe Densest k-Subgraph formally.

**Densest k-Subgraph**: Given a graph \( G = (V, E) \) and integer \( k \), find a subset \( S \subseteq V \) of \( k \) nodes to maximize the edges in the induced graph \( G[S] \).

The current best approximation for Densest k-Subgraph is \( O(n^{1/4 + \epsilon}) \) [1]; note that an \( O(k) \)-approximation is trivial. On the other hand we can rule out a PTAS for Densest k-Subgraph only under the the assumption that NP \( \not\subseteq \cap_{\epsilon > 0} \text{BPTIME}(2^{n^\epsilon}) \) [6]. Moreover [2] shows polynomial-factor integrality gaps for several strong SDP relaxations. Resolving the wide gap in the approximability of Densest k-Subgraph is a major open problem. There have been several problems that have been shown to be conditionally hard by giving a reduction from Densest k-Subgraph which has further cemented the importance of Densest k-Subgraph as a canonical problem.

To formally state the implication of our reduction it is more convenient to relate the approximation ratio to the parameter \( s \) which is the sum of the number of nodes and edges. The tight instances for the algorithm of [1] for Densest k-Subgraph have \( |E| = \Theta(|V|^{3/2}) \). For these instances, it is not known how to obtain an approximation ratio better than \( O(|V|^{1/4}) = O(s^{1/6}) \).

**Theorem 1.1** A polynomial-time \( \alpha(s) \) approximation algorithm for K-way Hypergraph Cut implies a polynomial-time \( O(\alpha^2(s + 1)) \)-approximation algorithm for Densest k-Subgraph.

When \( k \) is a fixed constant one can reduce K-way Hypergraph Cut and K-way SUB-MP to solving \( O(n^{k-1}) \) instances of the “terminal” version of these problems which have a \( 2(1 - 1/k) \) approximation. We refer the reader to [3, 4] for more details on these related problems.

## 2 Proof of Theorem 1.1

Let \( (G = (V, E), \ell) \) be an instance of Densest k-Subgraph, where to avoid confusion, we use \( \ell \) to denote the number of nodes in the subgraph we wish to find. We construct a hypergraph \( H = (A, \mathcal{F}) \) as follows. For each edge \( e \in E \) we create a node \( a_e \) and add it to \( U \). Moreover we add a new special node \( r \) to \( U \). Thus \( A = \{r\} \cup \{a_e \mid e \in E\} \). For each node \( v \in V \) we add a hyperedge \( f_v \) to \( \mathcal{F} \) where \( f_v = \{r\} \cup \{a_e \mid e \in \delta_G(v)\} \) where \( \delta_G(v) \) is the set of edges in \( E \) that are incident to \( v \) in \( G \). Thus \( H \) is basically the hypergraph obtained from \( G \) by flipping the role of nodes and edges and then adding the extra node \( r \) to each hyperedge. We also observe that \( |U| + |\mathcal{F}| = 1 + |V| + |E| = s + 1 \). The basic and simple claim about \( G \) and \( H \) is the following.
**Claim 2.1** For any $1 \leq \ell \leq |V|$, if there is set $S \subseteq V$ with $|S| = \ell$ and $|E_G(S)| = L - 1$ then the k-way Hypergraph Cut instance on $H$ with $k = L$ has a cut of value at most $\ell$. Moreover, given any $F \subseteq \mathcal{F}$ with $|F| = \ell'$ such that $H - F$ has $L'$ connected components then there is a subset $S' \subseteq V$ such that $|S'| = |F|$ and $|E_G(S')| = L' - 1$.

**Proof:** Consider a set $F \subseteq \mathcal{F}$ of hyperedges in $H$. Suppose we remove them from $H$. Let $V_F = \{v \in V \mid f_v \in F\}$ be the nodes in $G$ that correspond to the edges in $F$. Then a node $a_e \in A$ corresponding to an edge $e = uv$ is separated from $r$ in $H$ if both $u, v \in V_F$; in this case the node $a_e$ becomes an isolated node in $H - F$. Thus the number of connected components in $H - F$ is precisely equal to $|E_G(V_F)| + 1$. This correspondence proves both parts of the claim. $\square$

Suppose we have an $\alpha$ approximation for k-way Hypergraph Cut. We will obtain an $\alpha^2$-approximation for Densest k-Subgraph as follows. Let $(G, \ell)$ be a given instance of Densest k-Subgraph. First assume that we know the optimum solution value $L$ for the given instance. We construct $H$ the hypergraph as described and give $H$ and $k = L + 1$ to the $\alpha$-approximation algorithm for k-way Hypergraph Cut. By Claim 2.1 there is an optimum solution to the k-way Hypergraph Cut instance on $H$ of value $\ell$. Thus, the approximation algorithm will output a set $F \subseteq \mathcal{F}$ such that (i) $|F| \leq \alpha \cdot \ell$ and (ii) $H - F$ has at least $L + 1$ connected components. By the second part of the claim we can obtain a set $S' \subseteq V$ such that $|S'| \leq \alpha \cdot \ell$ and $|E_G(S')| \geq L$. A random subset $S$ of $S'$ where $|S| = \ell$ induces, in expectation, at least $L/\alpha^2$ edges. One can derandomize this step. Thus we can obtain a set $S \subseteq V$ such that $|S| = \ell$ and $|E_G(S)| \geq L/\alpha^2$. Since $L$ is the optimum value for the given instance of Densest k-Subgraph, we obtain the desired $\alpha^2$-approximation. We can remove the assumption of the knowledge of $L$ by trying all possible values of $L$ from 0 to $|E(G)|$. This finishes the proof of Theorem 1.1.

## 3 Open problems

The main open question is to obtain a hardness of approximation for k-way Hypergraph Cut under the $P \neq NP$ assumption. At this point we only have APX-Hardness coming from k-Cut; we should note that APX-Hardness for k-Cut has been claimed by Papadimitriou (see [8]) but as far as we know no published proof has appeared in the literature. k-way SUB-MP appears to be much more general than k-way Hypergraph Cut so it may be easier to first establish hardness for k-way SUB-MP. In fact it may be feasible to prove strong unconditional lower bounds for k-way SUB-MP in the oracle model via the techniques from [9, 4].

When $k$ is a fixed constant k-Cut can be solved in polynomial time [5]. k-way Hypergraph Cut is known to be solvable in polynomial time for $k \leq 3$ [10] but its status is open for any fixed $k \geq 4$. In fact we also do not know whether k-way Sym-SUB-MP or k-way SUB-MP are NP-Hard for any fixed $k > 2$.

## References


