Some Open Problems in Element-Connectivity

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Abstract

The notion of *element-connectivity* has found several important applications. An important tool for these applications is a graph reduction step that preserves element-connectivity [19, 4]. In this article we survey several open problems related to this reduction step as well as some related problems on edge-connectivity.

1 Introduction

Let $G = (V, E)$ be an undirected simple graph. Given two distinct nodes $u, v \in V$, we will use $\lambda_G(u, v)$ and $\kappa_G(u, v)$ to denote the edge-connectivity and vertex-connectivity between u and v in G respectively. Formally, these are defined to be the maximum number of edge-disjoint and internally node-disjoint paths between u and v in G. We say that G is k-edge-connected if $\lambda_G(u, v) \geq k$ for all distinct $u, v \in V$. Similarly G is k-vertex-connected if $\kappa_G(u, v) \geq k$ for all distinct $u, v \in V$.

Here we are concerned with a related connectivity measure. Let $T \subseteq V$ be a set of terminals; vertices in $V \setminus T$ are referred to as non-terminals. Some times we refer to the terminals as black nodes and the non-terminals as white nodes. For any two distinct terminals $u, v \in T$, element-connectivity between u and v is the maximum number of u-v paths in G that are pairwise "element"-disjoint where elements consist of edges and non-terminals. Note that element-disjoint paths need not be disjoint in terminals. We use $\kappa'_G(u, v)$ to denote the element-connectivity between u and v. If we sub-divide each edge between terminals via a new non-terminal then T forms an independent set and $\kappa'_G(u, v)$ is the maximum number of u-v paths that are disjoint in non-terminals. We use $\kappa'_G(T) = \min_{u,v \in T} \kappa'_G(u, v)$ to denote the global element-connectivity. Via Menger's theorem this is equal to the minimum number of elements whose deletion separates some pair of terminals.

Element-connectivity can be seen to generalize edge-connectivity by letting $T = V$. At the same time, element-connectivity is also closely related to vertex-connectivity. In particular, if T contains exactly two vertices s and t, then $\kappa'_G(s, t)$ is equal to $\kappa_G(s, t)$, the vertex-connectivity between s and t. Element-connectivity has found several applications. A graph reduction operation suggested by Hind and Oellermann [19] plays a key role in these. We set up some notation to state their reduction step. Let G/pq denote the graph obtained from G by contracting the edge pq, and $G - pq$ to denote the graph obtained by deleting pq.

Theorem 1.1 (Hind & Oellermann [19]). *Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ be a terminal-set such that $\kappa'_G(T) \geq k$. Let pq be* any *edge where $p, q \in V \setminus T$. Then $\kappa'_{G_1}(T) \geq k$ or $\kappa'_{G_2}(T) \geq k$ where $G_1 = G - pq$ and $G_2 = G/pq$.*

Chekuri and Korula generalized the theorem to show that the same reduction operation also preserves the *local* element-connectivity of every pair of terminals.

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Theorem 1.2 (Chekuri & Korula [4]). *Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ be a terminal-set. Let pq be* any *edge where $p, q \in V \setminus T$ and let $G_1 = G - pq$ and $G_2 = G/pq$. Then one of the following holds: (i) $\forall u, v \in T$, $\kappa'_{G_1}(u, v) = \kappa'_G(u, v)$ (ii) $\forall u, v \in T$, $\kappa'_{G_2}(u, v) = \kappa'_G(u, v)$.*

We refer to the preceding theorem as the reduction lemma following the usage from [4]. By repeatedly applying the reduction lemma, one obtains the following corollary.

Corollary 1.3. *Given a graph $G = (V, E)$ and a terminal set $T \subseteq V$ there is a minor $H = (V', E')$ of G such that (i) $T \subseteq V'$ and (ii) $V' \setminus T$ is an independent set in H and (iii) $\kappa'_H(u, v) = \kappa'_G(u, v)$ for all $u, v \in T$. In particular, if T is an independent set in G then H is a bipartite graph with bipartition $(T, V' \setminus T)$.*

The minor H in the previous corollary is called a *reduced graph* of G. A graph is *reduced* if there are no edges between non-terminals. A reduced graph $G = (V, E)$ where the terminals T form an independent set can be interpreted as a hypergraph $H = (T, E')$. H contains an edge e_v for every non-terminal v in G, where e_v is the set of neighbors of v in G. Element-connectivity of terminals T in G is equivalent to hypergraph edge-connectivity in H. It is easy to see from the hypergraph representation that κ'_G is induced by a symmetric submodular set function on T. Thus we have a Gomory-Hu tree representation for κ'_G and there are only $|T| - 1$ distinct element connectivity values.

We will also be discussing some edge-connectivity problems and we would be remiss not to mention the famous splitting-off theorem of Mader [33] which generalized an earlier theorem of Lovász [32].

Theorem 1.4 (Mader [33]). *Let $G = (V \cup \{s\}, E)$ be an undirected multi-graph, where $deg(s) \neq 3$ and s is not incident to a cut edge of G. Then s has two neighbours u and v such that the graph G' obtained from G by replacing su and sv by uv satisfies $\lambda_{G'}(x, y) = \lambda_G(x, y)$ for all $x, y \in V \setminus \{s\}$.*

The rest of the article has two sections. Section 2 discusses packing edge and element-disjoint Steiner trees and forests and open problems related to structural results in those topics. Section 3 discusses open problem on algorithmic issues that arise in element-connectivity.

2 Packing disjoint Steiner trees and forests

In this section we consider the problem of packing disjoint Steiner trees and forests. Let $G = (V, E)$ be an undirected simple graph and let $T \subseteq V$. A tree in G that contains all the terminals T is a T-tree or a Steiner tree on T. Given disjoint terminal sets $T_1, T_2, \ldots, T_h$ a Steiner forest for them is a forest in which each T_i is connected. We are interested in packing a maximum number of edge-disjoint or internally node-disjoint Steiner trees and Steiner forests. For the most part we will consider simple graphs. We say that an instance is *capacitated* if edges or nodes have non-negative integer-valued capacities. Capacitated problems in general graphs can often be reduced to uncapacitated ones in pseudo-polynomial time. However, when working with restricted families of graphs such as planar graphs, the reduction does not preserve planarity. Although we are primarily interested in integer packings we some times mention *fractional* packing results. Fractional packing results have close connections to approximation and integrality gap results for corresponding optimization results. To be concrete, an α-approximation for minimum-cost Steiner tree problem implies an α-approximation for fractional packing of Steiner trees. Whether the converse is true is an interesting open question that was raised in [21] and stated below. We refer the reader to [2, 21, 8] for results and discussion on this topic.

Problem 1. *Does an α-appproximation for fractionally packing Steiner trees in an edge-capacitated graph imply an α-approximation for the minimum-cost Steiner tree problem?*

We have two classical theorem on packing when $|T| = 2$ and when $T = V$.

Theorem 2.1 (Menger)**.** *The maximum number of edge-disjoint s-t paths in an undirected graph G is equal to the minimum s-t edge cut in G. The maximum number of internally node-disjoint s-t paths in an undirected graph G is equal to the minimum s-t node cut in G.*

Let $\mathcal{P}$ be the set of all partitions of the vertex set V of a graph. For a given $P \in \mathcal{P}$ let $E(P)$ denote the set of edges which cross the partition P, that is, each edge in $E(P)$ has its end points in different parts of P. The partition connectivity of G is $\min_{P \in \mathcal{P}} \lfloor \frac{|E(P)|}{|P|-1} \rfloor$ where $|P|$ is the number of parts in P.

Theorem 2.2 (Tutte & Nash-Williams)**.** *The maximum number of edge-disjoint spanning trees in a graph G is equal to the partition connectivity of G. More generally the maximum number of disjoint bases in a matroid $\mathcal{M}$ on ground set E with rank function r is given by $\lfloor \min_{S \subseteq E} \frac{|E \setminus S|}{r(E)-r(S)} \rfloor$.*

Corollary 2.3. *If G is $2k$-edge-connected then there are k edge-disjoint spanning trees in G.*

Frank, Király and Kriesell [14] generalized the theorem on packing edge-disjoint spanning trees in a graph to hypergraphs as follows. As in graphs a hypergraph $H = (V, E)$ is k-partition connected if $|E(P)| \geq k(|P| - 1)$ for all partitions $P \subseteq \mathcal{P}$ of the vertex set V. Here $E(P)$ is the set of hyperedges that cross the partition P.

Theorem 2.4 (Frank, Király & Kriesell [14])**.** *If a hypergraph $H = (V, E)$ is k-partition-connected then there are k-hyperedge-disjoint spanning subgraphs.*

Note that partition connectivity is a much stronger notion than connectivity in the context of hypergraphs; in fact a hypergraph can have k disjoint spanning subgraphs and may not be k-partition-connected. Nevertheless, Theorem 2.4 is helpful in various settings; we will see an application later on. We also note that the preceding theorem is derived via the matroid base packing theorem and a matroid defined on hypergraphs called the hypergraphic matroid due to Lorea [31].

When $|T| \geq 3$ there is no known exact characterization of the maximum number of disjoint Steiner trees. It is easy to see that if there are k-edge-disjoint T-trees in G then T is k-edge-connected. Similarly if there are k internally node-disjoint T-trees in G then T is k-element-connected in G. Thus edge and element connectivity upper bound the maximum number of edge-disjoint and element-disjoint T-trees respectively. In fact they also upper bound the fractional packing number. Thus, it makes sense to ask how close is the connectivity to the packing number.

2.1 Edge disjoint packing

Kriesell made the following conjecture for edge-disjoint T-trees.

Conjecture 1 (Kriesell [27])**.** *If T is $2k$-edge-connected in G then there are k-edge-disjoint T-trees in G.*

There are examples where $2k$-edge-connectivity is needed to guarantee k edge-disjoint T-trees. Lau [30] was the first to obtain a non-trivial result regarding Kriesell's conjecture. He proved that $24k$-connectivity for T suffices to obtain k edge-disjoint T trees. West and Wu [37] improved this to $6.5k$ and more recently DeVos, McDonald and Pivotto [10] obtained a bound of $(5k + 4)$. These results use Mader's splitting-off theorem as a starting point and several other tools. From splitting-off one can easily obtain a $1/2$-integral packing of k trees. Interestingly [10] uses some fractional packing results along the way to their result.

A stronger notion of Steiner-tree called a T-connector can be defined. A T-connector is a Steiner tree for T where each non-terminal node has even-degree. If this is the case then, via splitting off at the non-terminals, a T-connector can be mapped to a tree $R = (T, F)$ on T where the edges of F correspond to edge-disjoint paths in G. West and Wu [37] show that if T is $10k$ edge-connected then there are k edge-disjoint T-connectors. DeVos *et al.* [10] showed that $(6k+6)$-connectivity suffices.

West and Wu conjecture the following.

Conjecture 2 (West and Wu [37])**.** *If T is $3k$-edge-connected in G then there are k-edge-disjoint T-connectors.*

$3k$ is necessary in the preceding conjecture. The following special case of the preceding conjecture is open.

Conjecture 3 (Wu)**.** *If there are 3 edge-disjoint T-trees in G then G has a T-connector.*

One of the difficulties in working with T-connectors appears to be algorithmic.

Problem 2. *Is there a polynomial-time algorithm that given G and T decides whether there is a T-connector in G?*

Also of interest is to find an algorithm to find a minimum-cost T-connector in a given edge-weighted graph. This could be related to ideas in [20].

Steiner forest packing: For Steiner forests Chekuri and Shepherd [3] showed that if each T_i is $2k$-edge-connected then there is a $\frac{1}{2}$-integral packing of k Steiner forests. Lau [29, 28] showed that if each T_i is $30k$-edge-connected then there are k edge-disjoint Steiner forests. The following generalization of Kriesell's conjecture does not appear to have been explicitly stated previously but has been implicitly considered in prior work.

Conjecture 4. *If each T_i is $2k$-edge-connected then there are k-edge-disjoint Steiner forests.*

Lau's work on Steiner forests is based on extending his work on Steiner tree packing. We believe that subsequent work that obtained improved results for Steiner tree packing [37, 10] should lead to improved results for Steiner forest packing as well.

2.2 Element-disjoint packing

Hind and Oellerman's main motivation for considering element-connectivity and the reduction step was to address the packing of element-disjoint T-trees[1]. We observe that the reduction step recasts the problem of packing element-disjoint T-trees into the problem of packing edge-disjoint spanning sub-graphs in *hypergraphs*.

Hind and Oellerman considered the case when $|T| = 3, 4$ and proved that if T is k-element-connected then there are $\lfloor \frac{1}{|T|-1} \lceil \frac{|T|k}{2} \rceil \rfloor$ element-disjoint Steiner trees. Cheriyan and Salavatipour [6] proved the following theorem for the general case.

Theorem 2.5 (Cheriyan & Salavatipour [6])**.** *Let T be k-element-connected in G. Then there is a randomized polynomial-time algorithm that outputs $\Omega(k/\log n)$ element-disjoint T-trees with high-probability. Moreover there are examples where the number of element-disjoint T-trees is $O(k/\log n)$.*

The algorithm achieving the above is extremely simple. Viewed in the context of finding disjoint spanning sub-graphs in a hypergraph, the algorithm colors each hyperedge uniformly at random from colors 1 to $ck/\log n$ for some constant c and shows that each color class forms a connected sub-graph with high-probability.

The preceding result was refined and put in a more general context in [8]. We describe this connection in some detail.

[1] We use element-disjoint in place of internally node-disjoint for sake of convenience and to be consistent with some previous papers.

Packing edge-disjoint spanning trees in a graph is a special case of packing bases of a matroid and Theorem 2.2 gives a tight min-max theorem. Given a hypergraph $H = (V, E)$ we can cast the problem of finding hyperedge-disjoint spanning subgraphs as packing full-rank subsets[2] of a polymatroid as follows. By a polymatroid we refer to an integer-valued non-negative monotone submodular function. Given a polymatroid $f : 2^N \to \mathbb{R}_+$ over a ground set N we define $A \subseteq E$ to be a full-rank set of f if $f(A) = f(E)$. Note that rank functions of matroids are polymatroids with the property that $f(i) \leq 1$ for each $i \in N$. In the setting of hypergraphs we can define $f(A)$ as $|V| - \text{conn}(A)$ where $\text{conn}(A)$ is the number of connected components induced by edges in A.

Given an arbitrary polymatroid f we can ask how many disjoint full-rank subsets there are. A natural upper bound for this is given by $k^* = \lfloor \min_{S \subseteq E} \frac{\sum_{i \in N \setminus S} f_S(i)}{f_S(E)} \rfloor$. The notation $f_A(B)$ refers to $f(B) - f(A)$. Calinescu *et al.* [8] consider the problem of packing full-rank subsets in a polymatroid and showed that there are $(1 - o(1))k^*/\ln f(E)$ such sets. There are examples where this is tight. Moreover, for any fixed $\varepsilon > 0$, it is also NP-Hard to find more than $(1 - \varepsilon)k^*/\ln f(E)$ bases.

The random coloring of Cheriyan and Salavatipour can be extended to packing full-rank sets in a direct fashion. [8] gives a slightly different algorithm based on random permutations instead of random coloring which allows the algorithm to be derandomized via the use of min-wise independent permutations. Note that the bound obtained via this analysis on the number of element-disjoint Steiner trees is $(1 - o(1))k/\ln |T|$ which slightly improves the bound from [6]. There are examples where this is the right upper bound. Moreover it is known that finding two edge-disjoint connected sub hypergraphs is NP-Hard.

We now raise a couple of open problems.

Problem 3. *Given a hypergraph $H = (V, E)$ that is k-edge-connected are there $\Omega(k/\log d)$ edge-disjoint connected hypergraphs where d is the maximum size of any hyperedge?*

Theorem 2.4 implies that there are $\lfloor k/d \rfloor$ connected hypergraphs. The preceding problem is a special case of a more general problem on packing full-rank sets of a polymatroid that was raised in [8].

Problem 4. *Let $f : 2^E \to \mathbb{R}_+$ be a polymatroid and let $k^* = \lfloor \min_{S \subseteq E} \frac{\sum_{i \in N \setminus S} f_S(i)}{f_S(E)} \rfloor$ be the natural upper bound on the number of disjoint bases. Are there $\Omega(k^*/d)$ disjoint full-rank subsets for f where $d = \max_{e \in E} f(e)$? Are there $\Omega(k^*/d)$ disjoint full-rank subsets?*

One of the reasons we ask the preceding problems is that fractionally one can indeed pack $\Omega(k^*/\log d)$ disjoint bases via the result of Wolsey on minimum-cost submodular set cover [38].

Restricted graph families: Next we move to Steiner tree packing problem when the input graph comes from a restricted family of graphs. Aazami, Cheriyan and Jampani [1] used Theorem 2.4 to show that if G is a planar graph and T is k-element-connected in G then there are $(\lfloor \frac{k}{2} \rfloor - 1)$ element-disjoint T-trees. More generally they showed that there are $k/c_{\mathcal{G}}$ element-disjoint T-trees if G belongs to a proper minor-closed family of graphs $\mathcal{G}$ where $c_{\mathcal{G}}$ is a fixed constant that depends only on $\mathcal{G}$.

Conjecture 5 (Aazami, Cheriyan & Jampani)**.** *If T is k-element-connected in a planar graph G then there are $\lfloor \frac{k}{2} \rfloor$-element-disjoint T-trees.*

Our main question in this context is whether the improved results for special classes of graphs carry over to the capacitated setting? This is motivated by the fact that this is indeed true for the fractional setting via results on minimum-cost node-weighted Steiner tree problem [9].

Problem 5. *Can the results in [1] be carried over to the capacitated setting?*

[2] Iin [8] full-rank sets are called bases of f. Here we depart from that terminology.

In fact there is a special case of the above problem which we think is important to address. The results of Aazami *et al.* are somewhat restricted in that they hold only if T is a full set of terminals. The following seems to be open.

Problem 6. *Let $G = (V, E)$ be a simple graph from a proper minor-closed family of graph $\mathcal{G}$. Let $T \subset V$ be the terminal sets. Suppose $T' \subset T$ such that T' is k-element-connected in G. Is there a constant $c_{\mathcal{G}}$ that depends only $\mathcal{G}$ such that there $k/c_{\mathcal{G}}$-element-disjoint T'-trees in G?*

For planar graphs and graphs embedded on an surface a grid-expansion trick from [4] allows one to assume that $T' = T$ but it does not apply for a general minor-closed family of graphs. We raise Problem 6 for two reasons. First, it arises in the context of packing Steiner forests that we discuss further below. Second, one notices that the result of Aazami *et al.* comes close to resolving Kriesell's conjecture in planar graphs since packing element-disjoint Steiner trees is more general than packing edge-disjoint Steiner trees. However the reduction of packing edge-disjoint Steiner trees to that of packing element-disjoint case creates new terminals (one for each non-terminal node of the original graph) that are not necessarily well-connected to the original terminals. Aazami *et al.* lose a factor of 2 in going from element-connectivity to edge-connectivity; they show that if T is k-edge-connected in a planar graph then there are $\lfloor k/4 \rfloor$ edge-disjoint T-trees. On the other hand the approach from [4] allows one to obtain the same bound as in the element-connectivity case. This fact does not seem to have been noticed or explicitly mentioned previously.

Steiner forest packing: We now consider packing element-disjoint Steiner forests. Here, Chekuri and Korula [4] showed that if each T_i is k-element-connected then there are $\Omega(k/(\log h \log |T|))$-element-disjoint Steiner forests where $T = \cup_i T_i$.

Conjecture 6 (Chekuri & Korula)**.** *If each T_i is k-element-connected in G then there are $\Omega(k/\log |T|)$ element-disjoint Steiner forests.*

We note that fractionally one can pack $\Omega(k/\log |T|)$ element-disjoint Steiner forests; follows from [25].

Inspired by a question of Joseph Cheriyan and [1], Chekuri and Korula considered Steiner forest packing in restricted families of graphs. Unlike [1] which relied on Theorem 2.4, [4] developed a different approach. Using this approach they showed that in planar graphs there are $\lfloor k/5 \rfloor - 1$ Steiner forests if each T_i is k-element-connected. Their approach extends to graphs with bounded genus and bounded treewidth but the case of proper minor-closed graphs seems to be difficult.

Conjecture 7 (Chekuri & Korula)**.** *Let $G = (V, E)$ be a simple graph from a proper minor-closed family of graph $\mathcal{G}$. Let $T_1, \ldots, T_h \subset V$ be disjoint terminal sets and let $T = \cup_i T_i$. Suppose each T_i is k-element-connected in G, then there $k/c_{\mathcal{G}}$-element-disjoint Steiner forests in G where $c_{\mathcal{G}}$ is a constant that depends only $\mathcal{G}$. Moreover this extends to the capacitated case.*

We remark that a positive resolution to Problem 6 is necessary for a positive resolution to the preceding conjecture. In the case of planar graphs it is possible to extend Conjecture 5 to the Steiner forest case.

Conjecture 8. *If each T_i is k-element-connected in a planar graph G then there are $\lfloor \frac{k}{2} \rfloor$-element-disjoint Steiner forests.*

We summarize the known results that we discussed in a table along with conjectured bounds.

3 Algorithmic Aspects

In this section we consider algorithmic complexity of element-connectivity. Corresponding questions for the classical notions of edge and vertex-connectivity have been extensively studied while element-connectivity has

	General Graphs	Planar Graphs	Proper minor-closed
Edge-disj trees	$5k+4$ [10], $(2k)$	$2k+2$ [1, 4], $(2k)$	$5k+4$ [10], $(2k)$
Edge-disj forests	$30k$ [28], $(2k)$	$5k+1$ [4], $(2k)$	$30k$ [28], $(2k)$
Elem-disj trees	$(1+o(1))k\ln\|T\|$ [6, 8]	$2k+2$ [1], $(2k)$	$O(k)$ [1]
Elem-disj forests	$O(k\ln\|T\|\ln h)$ [4], $(O(k\ln\|T\|))$	$5k+1$ [4], $(2k)$	$O(k\ln\|T\|\ln h)$ [4], $(O(k))$

Figure 1: Known connectivity requirement for guaranteeing k disjoint Steiner trees or forests in various settings. The bounds in parenthesis are conjectured.

not been explored very much.

Some basic questions that one can ask are the following. In each of these question we assume that the input consists of an undirected graph $G=(V,E)$ with $n=|V|$ and $m=|E|$, a terminal set $T\subseteq V$ with $|T|=t$.

- How fast can $\kappa'_G(u,v)$ be computed for $u,v\in T$? This is the single-pair case.
- How fast can $\kappa'_G(T)$, the global elem-connectivity, be computed?
- How fast can the all-pair element-connectivity be computed?
- How fast can the reduced graph be obtained from G?

The recent paper [5] examined the above questions and obtained several results that are summarized in Fig 2. The entries for edge and vertex connectivity are shown as well.

	Min Cut	Min Cut(WHP)	All-pair	All-pair (WHP)	Reduce
λ	$\tilde{O}(m)$[23]	$\tilde{O}(m)$[22]	$\tilde{O}(n^{27/8})$[11]	$\tilde{O}(nm)$[17]	-
κ'	$O(\|T\|\ \mathrm{MF}(n,m))$	same as all-pair	$O(\|T\|\ \mathrm{MF}(n,m))$	$O(m^{\omega})$[7]	$O(\|T\|nm)$
κ	$O(n^{7/4}m)$[15]	$\tilde{O}(nm)$[18]	$O(n^{9/2})$[13]	$\tilde{O}(n^{2+\omega})$[16]	-

Figure 2: The running time for various connectivity problems. The row for κ' gives results from [5]. MF(n,m) is the running time for a maximum flow on unit capacity directed graph with n vertices and m edges. WHP indicates with high probability bounds for randomized algorithms. ω is the matrix multiplication constant. $\tilde{O}$ notation suppresses poly-logarithmic factors.

Open problems: Obvious open problems are to improve the run-times in the row for κ'. We highlight some of the more interesting ones by discussing some details of the algorithms. The run-time of $O(|T|\ \mathrm{MF}(n,m))$ for all-pair connectivity is obtained by computing the Gomory-Hu tree that represents κ'_G using $|T|-1$ single-pair computations (the graph on which the single-pair computations are done changes as part of the Gomory-Hu tree computation). For simplicity consider the case when $|T|$ is large, say $\Theta(n)$. Then the we obtain a run-time of $O(n\mathrm{MF}(n,m))$ which is significantly worse than $O(nm)$ even with recent dramatic improvements in MF(n,m). The fastest algorithm we have to compute the global element connectivity is via the all-pair problem. We observe that the seemingly harder global vertex-connectivity problems can be solved in randomized $\tilde{O}(nm)$ time [18].

Problem 7. *Obtain an $\tilde{O}(nm)$-time or better (randomized) algorithm for global element-connectivity.*

For unit-capacity graphs there is a randomized $\tilde{O}(nm)$ time algorithm to compute a Gomory-Hu tree for edge-connectivity [17]. At this point we recall that the reduced graph is a hypergraph. Thus connectivity problems in hypergraphs are a special case of computing element connectivity. For a hypergraph we use m to denote the number of edges in its bipartite representation which corresponds to treating it as an element connectivity problem; we use n to refer to the number of nodes in the hypergraph. We note that there is an $O(nm + n^2 \log n)$-time algorithm for computing the global min-cut of a hypergraph [26].

Problem 8. *Obtain an $\tilde{O}(nm)$-time (randomized) algorithm for global element-connectivity.*

Problem 9. *Obtain a faster than $\tilde{O}(nm)$-time (randomized) algorithm for global connectivity in a hypergraph.*

Problem 10. *Obtain a $\tilde{O}(nm)$-time (randomized) algorithm for computing a Gomory-Hu tree representation of the edge-connectivity of a hypergraph.*

Approximation: Recent breakthrough work has obtained near-linear time algorithms to compute a $(1-\varepsilon)$-approximation to the the maximum s-t flow in undirected capacitated graphs [36, 24, 35]. In particular this implies that one can compute $\lambda_G(s,t)$ to within an $(1-\varepsilon)$-factor in near-linear time. In this context it is worth noting that Matuala in 1993 [34] obtained a deterministic linear-time $(2+\varepsilon)$-approximation to the global edge-connectivity problem; it is only recently that a deterministic $\tilde{O}(m)$-time algorithm for global edge-connectivity problem in simple graphs has been obtained by Kawarabayashi and Thorup [23]. Note that Karger's well-known $\tilde{O}(m)$-time algorithm is randomized but works for arbitrary capacities. We believe that fast approximations to connectivity problems is an interesting topic. It is possible to ask whether near-linear-time algorithms or algorithms faster than what we currently have are possible to obtain a constant factor approximation for several problems. We highlight a few below.

- Can the recent techniques for fast algorithms for undirected max-flow be generalized to handle vertex capacities? In particular, Is there a near-linear-time constant factor approximation for $\kappa_G(s,t)$? Is there such an algorithm for the global vertex connectivity?
- Is there a near-linear-time constant factor approximation for s-t connectivity in a hypergraph?
- Is there a near-linear-time constant factor approximation for the global mincut in a hypergraph? We note that the packing results that we discussed previously imply a randomized near-linear-time $O(\log n)$-approximation for this problem.

One may also ask if there is a near-linear time algorithm to compute an approximation to the s-t mincut in a directed graph. This is *not* the same as asking for an approximation to maximum flow from s to t. We may be able to compute the cut directly via other methods such as submodular function minimization; see [12] for such an algorithm.

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